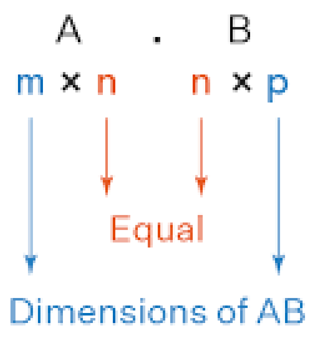
Week 12/13
Term 2
2024

HAWKER COLLEGE
Engage | Inspire | Achieve

MA1
Algebra and matrices

Goals



This week we are going to:

- perform matrix addition, subtraction, multiplication by a scalar, and matrix multiplication
- use matrices, including matrix products and powers of matrices, to model and solve problems; for example, costing or pricing problems, squaring a matrix to determine the number of ways pairs of people in a communication network can communicate with each other via a third person

Theoretical Components

Resources:
PDF file: Week 12-13 Notes and Exercises

The following clip gives a good introduction to matrix multiplication.
https://www.khanacademy.org/math/precalculus/precalc-matrices/multiplying-matrices-by-matrices/v/matrix-multiplication-intro

Knowledge Checklist

- Determining if matrix multiplication is possible
- Matrix multiplication
- Rules for matrix multiplication
- Applications of matrix multiplication

Order

1. Read through the notes and examples
2. Work through the exercises
3. Complete the investigation at the end of the booklet.
4. Complete the reflection at the end of the booklet
5. Come and see your teacher and make sure you are up to date.

Practical Components

Work through the exercises and show the completed tasks to your teacher.

Be sure to ask for help as needed for the successful completion of all tasks.

Remember to regularly check Google Classroom for messages.

Investigation

Other

WEEK 12-13 NOTES & EXERCISES

MATRIX MULTIPLICATION

Matrix multiplication is the multiplication of one matrix by another. It is not the same as scalar multiplication which is the multiplication of a matrix by a number. The matrix multiplication of two matrices *A* and *B* can be written as *A* × *B* or simply *AB*.

The method of multiplying matrices is shown in the example below.

Example

The number of CD's and DVD's sold by Fatima and Gaia are recorded in matrix *N*. The selling prices of the CD's and DVD's are shown in matrix *P*.

$$N = \begin{array}{c} \\ Fatima \\ Gaia \end{array} \begin{array}{c} \begin{matrix} CDs & DVDs \end{matrix} \\ \begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \end{array} \qquad P = \begin{array}{c} \\ CDs \\ DVDs \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 20 \\ 30 \end{bmatrix} \end{array}$$

We need to make a matrix, *S*, that shows the value of the sales made by each person.

Fatima sold: 7 CD's at \$20 + 4 DVD's at \$30

Gaia sold: 5 CD's at \$20 + 6 DVD's at \$30

$$S = \begin{array}{c} \\ Fatima \\ Gaia \end{array} \begin{array}{c} \$ \\ \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 4 \times 30 \end{bmatrix} \end{array}$$

The steps used in this multiplication follow the routine for the matrix multiplication of *N* × *P*.

As we move across the first row of matrix *N* we move down the column matrix *P*, adding the products of the pairs as we go.

Then we move across the second row of the matrix *N* and down the column of matrix *P* adding the products of the pairs of numbers as we go.

$$\begin{matrix} N & \times & P \end{matrix}$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ \square \quad \square \end{bmatrix}$$

$$\begin{bmatrix} 7 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 20 \\ 30 \end{bmatrix} = \begin{bmatrix} 7 \times 20 + 4 \times 30 \\ 5 \times 20 + 6 \times 30 \end{bmatrix}$$

$$= \begin{bmatrix} 140 + 120 \\ 100 + 180 \end{bmatrix}$$

$$= \begin{array}{c} \$ \\ \begin{bmatrix} 260 \\ 280 \end{bmatrix} \end{array} \begin{array}{c} \\ Fatima \\ Gaia \end{array}$$

Rules for matrix multiplication

- The number of columns in the first matrix must equal the number of rows in the second matrix.

If matrix **A** is of order $m \times n$ and matrix **B** is of order $n \times p$ then we can find the product **AB**.

$$m \times n \quad \times \quad n \times p$$

same

AB will be of order $m \times p$.

- To multiple two matrices, we multiply pairs of elements working across the rows in the first matrix and down the columns in the second matrix

Example

If we have $\mathbf{A} = \begin{bmatrix} 2 & 6 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 5 & 1 \\ 2 & 4 & 7 \end{bmatrix}$, and we need to calculate **AB**, we perform the following calculations by working across the rows in **A** and down the columns in **B**:

$$\begin{bmatrix} (2\times 3 \quad + \quad 6\times 2) & (2\times 5 \quad + \quad 6\times 4) & (2\times 1 \quad + \quad 6\times 7) \\ \mathbf{A} \text{ row } 1\times \mathbf{B} \text{ column } 1 & \mathbf{A} \text{ row } 1\times \mathbf{B} \text{ column } 2 & \mathbf{A} \text{ row } 1\times \mathbf{B} \text{ column } 3 \\ (0\times 3 \quad + \quad 1\times 2) & (0\times 5 \quad + \quad 1\times 4) & (0\times 1 \quad + \quad 1\times 7) \\ \mathbf{A} \text{ row } 2\times \mathbf{B} \text{ column } 1 & \mathbf{A} \text{ row } 2\times \mathbf{B} \text{ column } 2 & \mathbf{A} \text{ row } 2\times \mathbf{B} \text{ column } 3 \end{bmatrix}$$

When we multiply **A** row 1 by **B** column 3, the result goes in row 1, column 3 of the answer matrix.

So $\mathbf{AB} = \begin{bmatrix} 18 & 34 & 44 \\ 2 & 4 & 7 \end{bmatrix}$.

In general, to compute the (i, j)-element in the matrix AB, we multiply the elements of the ith row in A with the elements in the jth column of B, and sum all the products. For a pair of 2×2 matrices, this looks like:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Example

For the following matrices: $A = \begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 8 \\ 9 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 4 & 7 \end{bmatrix}$ $D = \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix}$

i. Decide if matrix multiplication can occur ie defined
ii. If it can occur do the matrix multiplication.

a) ***AB***   *A*   *B*

Write the order of each matrix.   3 × 2   2 × 1

The inside numbers are the same (both 2), so matrix multiplication is defined.

The outside numbers give the order. The order of the product of *AB* is 3 × 1.

Step 1

row 1 x column 1

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ \\ \\ \end{bmatrix}$$

Step 2

row 2 x column 1

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \end{bmatrix}$$

Step 3

row 3 x column 1

$$\begin{bmatrix} 5 & 2 \\ 4 & 6 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \times 8 + 2 \times 9 \\ 4 \times 8 + 6 \times 9 \\ 1 \times 8 + 3 \times 9 \end{bmatrix}$$

Which gives

$$= \begin{bmatrix} 40 + 18 \\ 32 + 54 \\ 8 + 27 \end{bmatrix}$$

So *A* ×*B*

$$= \begin{bmatrix} 58 \\ 86 \\ 35 \end{bmatrix}$$

b) ***BA***   *B*   *A*

Write the order of each matrix   2 × 1   3 × 2

The inside numbers are not the same **thus multiplication is not defined for *B* × *A***

c) ***CD***   *C*   *D*

Write the order of each matrix   1 × 3   3 × 1

Multiplication is defined. The order of *CD* will be 1 × 1

Move across the row of *C* and down the column of *D*, adding the products of the pairs.

$$\begin{bmatrix} 2 & 4 & 7 \end{bmatrix} \begin{bmatrix} 8 \\ 6 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \times 8 + 4 \times 6 + 7 \times 5 \end{bmatrix}$$

So *C* × *D*   =   [75]

EXERCISE 1

1. Carry out the matrix multiplication for the following pair of matrices.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

2. Tim made a mistake when he was asked to square matrix $\boldsymbol{M} = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$. His calculations are shown below. Tim has not written his final answer yet.

$$M^2 = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \times 2 + 1 \times 2 & 2 \times 1 + 1 \times 1 \\ 2 \times 2 + 0 \times 2 & 2 \times 1 + 0 \times 0 \end{bmatrix}$$
$$= \begin{bmatrix} \quad \end{bmatrix}$$

a Identify which element has the miscalculation.

b Write what Tim's final answer would have been.

c Write the correct answer for $\mathbf{M}^2$.

3. Determine the product: $A = \begin{bmatrix} 7 \\ 2 \\ 9 \end{bmatrix}$ $B = \begin{bmatrix} 10 & 15 \end{bmatrix}$

4. For the following matrices:

$$A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 3 \\ 0 & 8 \\ 2 & -5 \end{bmatrix}$$

i. Decide if matrix multiplication is possible
ii. If matrix multiplication is defined, give the order of the matrix multiplication and carry out the multiplication.

a. *AB*=

b. *BA*=

c. *CB*=

d. *BC* =

e. *AC*=

5. Determine the product PQ if $P = \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 4 & -2 \end{bmatrix}$ and $Q = \begin{bmatrix} -1 & 3 & 5 \\ 1 & 2 & -3 \end{bmatrix}$

6. If A= $\begin{bmatrix} \mathbf{3} & \mathbf{5} \\ \mathbf{5} & \mathbf{1} \end{bmatrix}$, then calculate $\boldsymbol{A^2}$. [Hint $A^2 = A \times A$]

7. If P = $\begin{bmatrix} \mathbf{8} & \mathbf{2} \\ \mathbf{4} & \mathbf{7} \end{bmatrix}$, Calculate the value of $\boldsymbol{P^2}$

8. If matrices A, B and C are of order 2 × 2, 3 × 1 and 2 × 3 respectively, the product of which two matrices will give a 2 × 3 matrix? The order of the matrices also needs to be stated.

APPLICATIONS OF MATRICES

1. Dodgy Bros sell vans, utes and sedans. The average selling price for each type of vehicle is shown in the first table.

Type of vehicle	Selling price ($)
Vans	$4 000
Utes	$12 500
Sedans	$8 500

The second table shows the total number of vans, utes and sedans sold at Dodgy Bros in one month.

Type of vehicle	Number of sales
Vans	5
Utes	8
Sedans	4

a. Stan is the owner of Dodgy Bros and wants to determine the total amount of monthly sales. Explain how matrices could be used to help Stan determine the total amount, in dollars, of monthly sales.

b. Perform a matrix multiplication that finds the total amount of monthly sales.

c. Brian is Stan's brother and the accountant for Dodgy Bros. In finding the total amount of monthly sales, he performs the following matrix multiplication.

$$\begin{bmatrix} 5 \\ 8 \\ 4 \end{bmatrix} \begin{bmatrix} 4000 & 12\,500 & 8500 \end{bmatrix}$$

Explain why this matrix multiplication is not valid for this problem.

2. In an AFL game of football, 6 points are awarded for a goal and 1 point is awarded for a behind. St Kilda and Collingwood played two games, with the two results given by the following matrix multiplication.

$$\begin{bmatrix} 9 & 14 \\ 10 & 8 \\ 16 & 12 \\ 7 & 10 \end{bmatrix} \begin{bmatrix} 6 \\ 1 \end{bmatrix} = \begin{bmatrix} C_1 \\ S_1 \\ C_2 \\ S_2 \end{bmatrix}$$

Complete the matrix multiplication to determine the scores in the two games.

3. Evaluate the following matrix multiplications.

a. $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$

c. Using your results from parts a and b, when will AB be equal to BA?

d. If A and B are not of the same order, is it possible for AB to be equal to BA?

WEEK 12-13 INVESTIGATION

Secret codes - Encoding and decoding

History has many accounts of the vital role that codes have played in protecting sensitive information used in wars and conspiracies. In the 1570's Mary, Queen of Scots, sent encoded messages from prison to Catholic supporters who planned to overthrow the Protestant Queen Elizabeth (Queen of England). Elizabeth was reluctant to execute her cousin Mary without direct evidence linking her to the plot. The charges were laid by the Principal Secretary, Sir Francis Walsingham. Unfortunately for Mary, Walsingham was also England's spymaster. He used an expert to break the code, and Mary was duly beheaded on February 8th, 1587.

Perhaps the most famous is the so called 'Enigma' code. During the Second World War the Germans used a machine to encode their messages. Alan Turing, a British mathematician developed the 'Bombe', a machine which eventually led to the ability to decode the German messages. This greatly helped the Allied war effort.

In the past codes usually involved swapping letters in a message for other letters. This is called **encoding** the message. It must be done using some specific pattern so that the **encoded** message can later be **decoded** by the person receiving the message.

Suppose we think of each letter as a number:

A	B	C	D	E	F	G	H	I	J	K	L	M
1	2	3	4	5	6	7	8	9	10	11	12	13
N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

The weakness of this type of code is that in the English language, *E* is the most frequent occurring letter, followed by *T* and then *A*. A table of frequencies for letters can be used to replace numbers occurring in about the same frequency and hence break the code.

A more effective means of encoding makes use of <u>matrix multiplication.</u>

Example: To **encode** the following message: MATHS IS GREAT

Step 1: Write the message using a 2 x p matrix.

$$\begin{bmatrix} M & T & S & S & R & A \\ A & H & I & G & E & T \end{bmatrix}$$ *There are 12 letters so this will be a 2 x 6 matrix.*

Step 2: Replace each letter with its corresponding letter from the alphabet.

$$\begin{bmatrix} 13 & 20 & 19 & 19 & 18 & 1 \\ 1 & 8 & 9 & 7 & 5 & 20 \end{bmatrix}$$

Step 3: Encode the message by multiplying it by an encoding 2 x 2 matrix

For this example, the encoding matrix is $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} 13 & 20 & 19 & 19 & 18 & 1 \\ 1 & 8 & 9 & 7 & 5 & 20 \end{bmatrix} = \begin{bmatrix} 27 & 48 & 47 & 45 & 41 & 22 \\ 41 & 76 & 75 & 71 & 64 & 43 \end{bmatrix}$$

For numbers that are above 26, subtract 26 until the values are between 1 and 26. (For numbers that are below 1, add 26 until the values are between 1 and 26.)

$$\begin{bmatrix} 1 & 22 & 21 & 19 & 15 & 22 \\ 15 & 24 & 23 & 19 & 12 & 17 \end{bmatrix} \rightarrow \text{Convert to a letter} \begin{bmatrix} A & V & U & S & S & V \\ O & X & W & S & L & Q \end{bmatrix}$$

The encoded message is: AOVXUWSSSLVQ

To **decode** a message we need to use the inverse matrix.

The inverse matrix of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is given by $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

The decoder matrix in this example is $\frac{1}{2x2-1x3}\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$.

Note the values of a,b,c,d were chosen to that $ad - bc = 1$ which makes the inverse matrix simpler (no fractions)

To decode AOVXUWSSSLVQ, follow the same process as above but using the decoder matrix.

$$\begin{bmatrix} A & V & U & S & S & V \\ O & X & W & S & L & Q \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 22 & 21 & 19 & 15 & 22 \\ 15 & 24 & 23 & 19 & 12 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} .\times \begin{bmatrix} 1 & 22 & 21 & 19 & 15 & 22 \\ 15 & 24 & 23 & 19 & 12 & 17 \end{bmatrix} = \begin{bmatrix} -13 & 20 & 19 & 19 & 18 & 27 \\ 27 & -18 & -17 & -19 & -21 & -32 \end{bmatrix}$$

$$\begin{bmatrix} 13 & 20 & 19 & 19 & 18 & 1 \\ 1 & 8 & 9 & 7 & 5 & 20 \end{bmatrix} \rightarrow \begin{bmatrix} M & T & S & S & R & A \\ A & H & I & G & E & T \end{bmatrix} \rightarrow \text{MATHSISGREAT}$$

The encoded message has been decrypted to the original message! 😊

MATHS IS GREAT

Your task:

a) Write a message and encode it using an encoding matrix of your choice.

b) Give your encoded message and the encoding matrix to another person. *Include the name of this person so I can check your working.*

Person you gave message to:

Person you received message from:

Encoded message:

Encoding matrix:

Decode the message you were given. Show working.

MARKING RUBRIC

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Portfolio Task	Student completes the portfolio task of the brief to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and communications	Student responses are accurate and appropriate in presentation of mathematical ideas in different contexts, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and techniques	Student submitted work selects and applies appropriate mathematical modelling and problem solving techniques to solve practical problems, and demonstrates proficiency in the use of mathematical facts, techniques and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises and portfolio task by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
		FINAL			**/20**

Student Reflection:
How did you go with this week's work?

What was interesting?

What did you find easy?

What do you need to work on?