Week 7/8
Term 1
2024

HAWKER COLLEGE
Engage | Inspire | Achieve

MA3
Growth and decay in sequences

Goals

This fortnight we are going to:

- use recursion to generate a geometric sequence
- display the terms of a geometric sequence in both tabular and graphical form and demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations
- deduce a rule for the *n*th term of a particular geometric sequence from the pattern of the terms in the sequence, and use this rule to make predictions
- use geometric sequences to model and analyse (numerically, or graphically only) practical problems involving geometric growth and decay; for example, analysing a compound interest loan or investment, the growth of a bacterial population that doubles in size each hour, the decreasing height of the bounce of a ball at each bounce; or calculating the value of office furniture at the end of each year using the declining (reducing) balance method to depreciate.

Theoretical Components

Resources:
PDF file: Week 7/8 Notes & Exercises

What is a geometric sequence?
https://www.youtube.com/watch?v=1z8QKFFU3Hc
Sum to infinity and the concept of convergence
https://www.youtube.com/watch?v=PSA6mr0oLzk

Geometric progressions:

$$t_n = ar^{n-1}$$

Sum of a geometric series:

$$S_n = \frac{a(r^n-1)}{r-1};\ r \neq 1$$

Sum to infinity:

$$S_\infty = \frac{a}{1-r}$$

Order

1. Read through the notes and examples
2. Work through the exercises
3. Complete the investigation at the end of the booklet.
4. Complete the reflection at the end of the booklet
5. Come and see your teacher and make sure you are up to date.

Practical Components

Work through the exercises and show the completed tasks to your teacher.

Be sure to ask for help as you need for the successful completion of all tasks.

Remember to regularly check Google Classroom for messages.

Knowledge Checklist

- Geometric sequences
- Terms of a geometric sequence
- Geometric series
- Infinite sum of a GS

Investigation

Complete the task at the end of the booklet and submit your work for checking. 😊

Quiz/forum/ other

In **Week 9** you are to sit an **In-Class Task** worth 20% (with your weekly investigations) in your first lesson for the week (Monday). It is an **"open book"** task given under test conditions. You are allowed to bring in any of your notes and worked exercises since Week 1 and, of course, your calculator. You will need to come prepared with your Chromebook/laptop.

MATHEMATICAL APPLICATIONS 3

WEEK 7/8 NOTES & EXERCISES

GEOMETRIC SEQUENCES

A farmer is breeding worms which he hopes to sell to the local shire councils for use in the decomposition of waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $1 + \frac{1}{10}$ or 1.1. He starts off with 10 kg of worms. By the beginning of the second week, he will expect $10 \times 1.1 = 11$kg of worms, by the start of the third week, he would expect $11 \times 1.1 = 10 \times 1.1^2 = 12.1$kg of worms, and so on. This is an example of a geometric sequence.

A geometric sequence is one whereby the first term is multiplied by a number, known as the common ratio, to create the second term which is multiplied by the common ratio to create the third, and so on. The first term in a geometric sequence is referred to as *a* and the common ration is referred to as *r*. consider the geometric sequence where $a = 1$ and $r = 3$. The terms in the sequence are: 1, 3, 9, 27, 81, …

To discover the common ratio, *r*, of a geometric sequence you need to calculate the ratio of successive terms, namely $\frac{t_2}{t_1}$. Alternatively, you could calculate $\frac{t_3}{t_2}$ or $\frac{t_4}{t_3}$.

EXERCISE 1

1. Which of the following are geometric sequences?
For the geometric sequences you have identified, find *a* and *r*.

a. 1, 2, 4, 8, 16, …

b. 2, 6, 18, 54, 162, …

c. 1, 4, 16, 64, 256, …

d. 2, 6, 12, 24, 48, …

e. 1, 5, 25, 100, 125, …

f. 0, 2, 4, 8, 16, …

g. -2, 4, -8, 16, -32, …

h. -1, -5, 10, -20, -40, …

i. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

FIND THE TERMS OF A GEOMETRIC SEQUENCE

Consider the finite geometric sequence of five terms for which $a = 3$ and $r = 4$.

3, 12, 48, 192, 768.

Now:

$t_1 = 3$	$t_1 = a$	
$t_2 = 3 \times 4$	$t_2 = a \times r$	$t_2 = a \times r^1$
$t_3 = 3 \times 4 \times 4$	$t_3 = a \times r \times r$	$t_3 = a \times r^2$
$t_4 = 3 \times 4 \times 4 \times 4$	$t_4 = a \times r \times r \times r$	$t_4 = a \times r^3$
$t_5 = 3 \times 4 \times 4 \times 4 \times 4$	$t_5 = a \times r \times r \times r \times r$	$t_5 = a \times r^4$

We should notice a pattern emerging. That pattern can be described by the equation:

$$t_n = 3 \times 4^{n-1}$$

For example, if $n = 5$, then $t_5 = 3 \times 4^4$

We can generalise this rule for all geometric sequences.

$$\boldsymbol{t_n = ar^{n-1}}$$

Where t_n is the *n*th term,
a is the first term,
r is the common ratio.

This rule enables us to find any term of a geometric sequence provided we know the value of *a* and *r*.

Example

Find the 12th term of the geometric sequence: 2, 10, 50, 250, 1250, …

Solution

Step 1: Find the value for *a*.

$$a = 2$$

Step 2: Find the value for *r* (if stated it is a geometric sequence).

$$r = \frac{10}{2} = 5$$

Step 3: Use the rule to find the 12th term.

$$t_{12} = 2 \times 5^{11} = 97{,}656{,}250$$

Step 4: Write your answer.

The 12th term is 97 656 250

Example

The 2nd term of a geometric sequence is 8 and the 5th term is 512. Find the 10th term of this sequence.

Solution

Step 1: We know that $t_2 = 8$ and that $t_n = ar^{n-1}$.

$t_2 = ar^1$
$t_2 = 8$

$\boldsymbol{ar^1 = 8} \quad equation\ 1$

Step 2: We know that $t_5 = 512$ and that $t_n = ar^{n-1}$.

$t_5 = ar^4$
$t_5 = 512$

$\boldsymbol{ar^4 = 512}\ equation\ 2$

Step 3: Solve the two equations simultaneously by eliminating *a*, to find *r*.

$$\frac{ar^4}{ar^1} = \frac{512}{8}$$

Divide equation 2 by equation 1

$r^3 = 64$
$r = 4$

Step 4: To find *a*, substitute the value of *r* into equation 1

$a \times 4 = 8$
$a = 2$

Step 5: Write down the rule.

$t_n = 2 \times 4^{n-1}$

Step 6: Find the 10th term, let $n = 10$

$t_{10} = 2 \times 4^9$
$t_{10} = 524{,}288$

Step 7: Write your answer

The 10th term in the sequence is 524 288

EXERCISE 2

1. Find the value of the term specified for the given geometric sequences.

a. Find the 15^{th} term of the geometric sequence 2, 8, 32,128, 512, …

b. Find the 10^{th} term of the geometric sequence 2, 12, 72, 432, 2592, …

c. Find the 20^{th} term of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

d. Find the 8^{th} term of the geometric sequence 3.1, 8.06, 20.956, 54.4856, 141.66256, …

e. Find the 9^{th} term of the geometric sequence -2, -8, -32, -128, -512, …

f. Find the 10^{th} term of the geometric sequence $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

2. Find the value of the term specified for the specified geometric sequences.

a. The 2nd term of a geometric sequence is 6 and the 5th term is 162. Find the 10th term.

b. The 2nd term of a geometric sequence is 6 and the 5th term is 48. Find the 12th term.

3. For the geometric sequence 3, *p*, *q*, 192, … find the values for *p* and q.

GEOMETRIC SERIES

When we add up or sum the terms in a sequence, we get the series for that sequence. If we look at the geometric sequence {2, 6, 18, 54, …} where the first term $a = 2$ and the common ratio is 3, the series becomes $2 + 6 + 18 + 54 + \cdots$

$$S_1 = t_1 = 2$$
$$S_2 = t_1 + t_2 = 2 + 6 = 8$$
$$S_3 = t_1 + t_2 + t_3 = 2 + 6 + 18 = 26$$
$$S_4 = t_1 + t_2 + t_3 + t_4 = 2 + 6 + 18 + 54 = 80$$

The sum of the first n terms of a geometric sequence is given by:

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1}$$

Where a is the first term of the sequence and r is the common ratio.

Example

Find the sum of the first 9 terms of the sequence 0.25, 0.5, 1, 2, 4, …

Solution

The series is geometric and $a = 0.25$

Find the value of r by testing ratios of the given terms.

$$\frac{t_2}{t_1} = \frac{0.5}{0.25} = 2$$
$$\frac{t_3}{t_2} = \frac{1}{0.5} = 2$$
$$\frac{t_4}{t_3} = \frac{2}{1} = 2$$
$$\frac{t_5}{t_4} = \frac{4}{2} = 2$$

Using $S_n = \frac{a(r^n-1)}{r-1}$ gives

$$S_9 = \frac{0.25(2^9 - 1)}{2 - 1}$$
$$S_9 = \frac{0.25(512 - 1)}{1}$$

$$S_9 = 127.75$$

Example

The third term of a geometric sequence is 11.25 and the sixth term is 303.75. Find the sum of the first 10 terms of the sequence correct to 1 decimal place.

Solution

We need to find a and r. We know that $t_3 = 11.25$ and that $t_n = ar^{n-1}$.	$t_3 = ar^2$ $t_3 = 11.25$ $ar^2 = 11.25$ *Equation 1*
We know that $t_6 = 303.75$ and that $t_n = ar^{n-1}$.	$t_6 = ar^5$ $t_6 = 303.75$ $ar^5 = 303.75$ *Equation 2*
Solve the two equations simultaneously by eliminating *a*, to find *r*.	$\frac{ar^5}{ar^2} = \frac{303.75}{11.25}$
Equation 2 divided by equation 1	$r^3 = 27$ $r = 3$
To find *a*, substitute the value of *r* into equation 1	$a \times 3^2 = 11.25$ $a = 1.25$
Since $r > 1$, use $S_n = \frac{a(r^n-1)}{r-1}$	$S_{10} = \frac{1.25(3^{10}-1)}{3-1}$ $S_{10} = 36905$
Write your answer	*The sum of the first ten terms of the geometric series is 36905*

EXERCISE 3

1. Find the sum of the following:

a. First 12 terms of the geometric sequence 2, 6, 18, 54, 162, …

b. First 7 terms of the geometric sequence 5, 35, 245, 1715, 12 005, …

c. First 15 terms of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

d. First 12 terms of the geometric sequence -0.1, -0.4, -1.6, -6.4, -25.6, …

e. First 11 terms of the geometric sequence 128, 64, 32, 16, 8, …

2. The second term of a geometric sequence is 10 and the fifth is 80. Find the sum of the first 12 terms of the sequence.

3. The second term of a geometric sequence is 6 and the fifth term is 48. Find the sum of the first 15 terms of the sequence.

4. How many terms of the geometric sequence 3, 6, 12, 24, 48, … are required for the sum to be greater than 3000? Use trial and error.

5. On the first day, Abbey hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.

a. Write the geometric sequence for the first five days of the above real-life situation.

b. Find the value of r.

c. How many people are told of the rumour on the 12^{th} day?

d. How many people altogether have heard the rumour on the 12^{th} day?

THE INFINITE SUM OF A GEOMETRIC SEQUENCE WHERE $r < 1$

A student stands at one side of a road, 10 metres wide, and walks half-way across. The student then walks half of the remaining distance across the road, then half the remaining distance again and so on. Will the student ever make it *past* the other side of the road? And does the width of the road affect your answer?

Here, we have a sequence $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \frac{5}{32}, \ldots$

The terms are getting smaller and smaller, in fact, no matter how many terms we add the sum cannot be greater than 10 (given that the road is only 10 m wide).

This type of scenario is called a 'sum to infinity'.

Consider the following sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$ Here $r = \frac{1}{2}$

As n increases the value of the term decreases. $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}, \left(\frac{1}{2}\right)^{20} = \frac{1}{1048576}$

As n gets larger, the term gets smaller. In fact, as n approaches infinity, the term approaches zero.

As $n \to \infty, r^n \to 0$
Note: This only occurs when r is a fraction between -1 and 1.
This is written as $-1 < r < 1$.

So, for very large values of n, our formula $S_n = \frac{a(1-r^n)}{1-r}$ becomes $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$.

The sum to infinity of a geometric sequence for which $-1 < r < 1$ is given by:

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

Example

Find the sum to infinity for the sequence $t_n: \{10,\ 1,\ 0.1, \ldots\}$

Solution

We know that the first term, a, is 10 and $r = 0.1$. $\quad a = 10, r = 0.1$

Write the formula for the sum to infinity. $\quad S_\infty = \frac{a}{1-r}; |r| < 1$

Substitute $a = 10$ and $r = 0.1$ into the formula and evaluate. $\quad S_\infty = \frac{10}{1-0.1} = \frac{10}{0.9} = \frac{100}{9}$

Thus, no matter how many terms we add the sum can never exceed $\frac{100}{9}$.

Example

Find the fourth term in the geometric sequence whose first term is 6 and whose sum to infinity is 10.

Solution

Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1-r}; \ \lvert r \rvert < 1$
From the question, it is known that the infinite sum is equal to 10 and that the first term a is 6. Write down this information.	$a = 6; S_\infty = 10$
Substitute known values into the formula. Solve for r.	$10 = \frac{6}{1-r}$ $10(1-r) = 6$ $10 - 10r = 6$ $-10r = -4$ $r = 0.4$
Write the general formula for the nth term of the geometric sequence.	$t_n = ar^{n-1}$
To find the fourth term, substitute $a = 6, n = 4$ and $r = 0.4$ into the formula and evaluate.	$t_4 = 6 \times 0.4^{4-1} = 0.384$

EXERCISE 4

1. Find the sum to infinity for the sequence:

a. $t_n: \left\{1, \frac{1}{2}, \frac{1}{4}, \ldots\right\}$

b. $t_n: \left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots\right\}$

c. $t_n: \left\{1, \frac{1}{3}, \frac{1}{9}, \ldots\right\}$

d. $t_n: \left\{1, \frac{2}{3}, \frac{4}{9}, \ldots\right\}$

2. For the infinite geometric sequence $\left\{\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots\right\}$. Find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

3. A sequence of numbers is defined by $t_n\colon \{9,\ -3,\ \ 1, \ldots\}$

a. Find the sum of the first 9 terms.

b. Find the sum to infinity, S_∞.

4. A sequence of numbers is defined by $t_n = 3 \times \left(\frac{1}{2}\right)^{n-1},\ n \in \{1,\ 2,\ 3, \ldots\}$.

a. List the first 5 terms of the sequence.

b. Find the sum of the first 20 terms.

c. Find the sum to infinity, S_∞

APPLICATIONS OF GEOMETRIC SEQUENCES AND SERIES

Example

A city produced 100 tonnes of rubbish in the year 2001. Forecasts suggest that this may increase by 2% each year. If these forecasts are true,

a) what will be the city's rubbish output in 2005?
b) in which year will the amount of rubbish reach 120 tonnes?
c) what was the total amount of rubbish produced by the city in the years 2001, 2002 and 2003?

Solution

This is an example of a geometric sequence where a = 100 and r = 1.02.

Note that r ≠ 0.02. If this was the case, then multiplying 100 by 0.02 would result in a lesser amount of rubbish in the second year and so on. We are told that the amount of rubbish increases by 2%. That is the original amount plus an extra 2%, or: original amount + 2% of original amount = original amount (1 + 2%) = original amount (1 + 0.02) = 1.02 × original amount.

a)

1. Find the first term, a.
2. Determine the common ratio, r.
3. Determine which term is represented by the amount of rubbish for the year 2005.
4. Use $t_n = ar^{n-1}$ to find the amount of rubbish collected in the fifth year.
5. Write your response.

a $a = 100$

Increase by 2%

$1 + 2\% = 1 + 0.02$

$r = 1.02$

Year 2001 is the first term, so $n = 1$.
Year 2002 is the second term, so $n = 2$.
Year 2005 is the fifth term, so $n = 5$.

$t_5 = 100 \times 1.02^{5-1}$
$= 100 \times 1.0824$
$= 108.24$

The amount of rubbish produced in the fifth year, or 2005, will be 108.24 tonnes.

b) Solve by using trial and error:

1. Use $t_n = ar^{n-1}$ and $t_n = 120$.
2. Try various values of n.
3. Write your answer.

$100(1.02)^{n-1} = 120$

$(1.02)^{n-1} = 1.2$

Let $n = 10$, $(1.02)^9 = 1.195$

Let $n = 11$, $(1.02)^{10} = 1.21$

During the 11th year, that is, during 2011, the rubbish will have exceeded 120 tonnes.

c)

1. We need to find the sum of the first 3 years.
Use $S_n = \frac{a(r^n - 1)}{r - 1}$ where $n = 3$.

c $S_3 = \frac{100(1.02^3 - 1)}{1.02 - 1}$

$= 306.04$

EXERCISE 5

1. A farmer harvests 4 tonnes of lucerne in his first year of production. In his business plan, he has estimated an annual increase of 6% on his harvest tonnage.

a. According to this plan, how many tonnes of lucerne should he harvest in his 7th year of production?
b. How much will he expect to harvest in the first three years?
c. In which year will his harvest reach 10 tonnes?

2. The population of a town is decreasing by 10% each year. The present population is 10000.

a. What will be the population in 5 years' time?
b. How long before the population will be zero?

3. A company exported $300 000 worth of manufactured goods in its first year of production. According to the business plan of the company, this amount should increase each year by 7.5%.

a. How much would the company be expected to export in its 5th year?
b. What is the total amount exported by the company in its first 7 years of operation?
c. In which year would exports exceed $500 000?

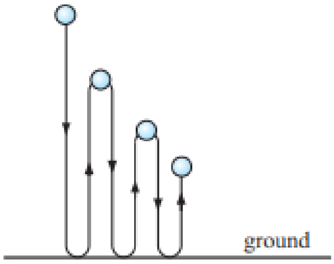
4. Recurring decimals can be expressed as rational numbers ie fractions. The technique for doing this is as follows.
0.22222222……. = 0.2 + 0.02 + 0.002 + 0.0002 + ………
This is a gemoemtric progression with $a = 0.2 \; and \; r = \frac{0.02}{0.2} = 0.1$. Using the sum to infinity $S_\infty = \frac{a}{1-r}$ gives $\frac{0.2}{1-0.1}$ gives $\frac{0.2}{0.9}$ which simplifies to $\frac{2}{9}$ (multiply numerator and denominator by 10).
Thus $\frac{2}{9}$ = 0.222222………
Use this technique to convert the recurring decimals to fractions.

a. 0.6666666 …

b. 0.777777 …

c. 0.27272727 …

5. A ball is dropped from a height of 1 metre. On each bounce it reaches a height of 80% of the previous bounce. What is the total distance the ball travels before it comes to rest?



6. Suppose you save $1 the first day of April, $2 the second day, $4 the third day, $8 the fourth day, and so on. That is, each day you save twice as much as you did the day before.

a. What will you put aside for savings on the last day of the month?
b. What will your total savings be for the month?

7. In a lake, there is a patch of lily pads. Every day, the patch doubles in size. If it takes 48 days for the patch to cover the entire lake, how long would it take for the patch to cover half of the lake?

AST STYLE QUESTION

"If a link is broken, the entire chain breaks" – Yiddish Proverb

> Hi!
> My name is Bill Weights, founder of Super Scooper Ice Cream. I am offering you a gift certificate for our signature "Super Bowl" (a $4.95 value) if you forward this letter to 10 people. When you have finished sending this letter to 10 people, a screen will come up. It will be your Super Bowl certificate. Print that screen out and bring it to your local Super Scooper Ice Cream store. The server will bring you the most wonderful ice cream creation in the world – a Super Bowl with three yummy ice cream flavours and three toppings!
>
> This is a sales promotion to get our name out to young people around the country. We believe this project can be a success, but only with your help. Thank you for your support
>
> Sincerely,
> Bill Weights
> Founder of Super Scooper Ice Cream

These chain emails rely on each person that receives the email to forward it on. Have you ever wondered how many people might receive the email if the chain remains unbroken? To figure this out, assume that it takes a day for the email to be opened, forwarded, and then received by the next person. On day 1, Bill Weights starts by sending the email out to his 10 closest friends. They each forward it to 10 people so that on day 2, it is received by 100 people. The chain continues unbroken.

Write the sequence.

Write the sequence as a power of 10.

How many people will receive the email on day *n*?

How many people will receive the email on day 7?

If Bill gives away a Super Bowl that costs $4.95 to every person that receives the email during the first week, how much will he have spent?

INVESTIGATION WEEK 7/8

There is a famous legend about the origin of chess which goes like this. When the inventor of the game showed it to an emperor of India, the emperor was so impressed by the new game, that he said to the inventor

"Name your reward!"

The man responded,

"Oh emperor, my wishes are simple. I only wish for this. Give me one grain of wheat for the first square of the chessboard, two grains for the next square, four for the next, eight for the next and so on for all 64 squares, with each square having double the number of grains as the square before."

The emperor agreed, amazed that the man had asked for such a small reward – or so he thought. He asked for a bag of wheat to be brought and assumed the man would soon be on his way. The bag of wheat was soon used up and the emperor now realised that he could never grant the reward as from then on, he would be doubling the grains of wheat on each square.

1. How many grains of wheat are needed for the 64^{th} square?
2. How many grains of wheat are needed altogether?
3. If a grain of wheat weighs 0.1 grams, how many kilograms of wheat does this represent? (1000 grams = 1 kilogram)
4. Compare this to the world's total production of wheat for 1 year.

Show how the answers are calculated (show your working out) but you may need to look up the answers as the numbers are very big. Express your final answers in scientific notation where necessary.

MARKING RUBRIC

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation	Student completes the investigation of the brief to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and communications	Student responses are accurate and appropriate in presentation of mathematical ideas in different contexts, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and techniques	Student submitted work selects and applies appropriate mathematical modelling and problem solving techniques to solve practical problems, and demonstrates proficiency in the use of mathematical facts, techniques and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
		FINAL			**/20**

Student Reflection:
How did you go with this week's work?

What was interesting?

What did you find easy?

What do you need to work on?