Week 5/6
Term 1
2024

HAWKER COLLEGE
Engage | Inspire | Achieve

MA3
Growth and decay in sequences

Goals

This fortnight we are going to:

- use recursion to generate an arithmetic sequence
- display the terms of an arithmetic sequence in both tabular and graphical form and demonstrate that arithmetic sequences can be used to model linear growth and decay in discrete situations
- deduce a rule for the *n*th term of a particular arithmetic sequence from the pattern of the terms in an arithmetic sequence, and use this rule to make predictions
- use arithmetic sequences to model and analyse practical situations involving linear growth or decay; for example, analysing a simple interest loan or investment, calculating a taxi fare based on the flag fall and the charge per kilometre, or calculating the value of an office photocopier at the end of each year using the straight-line method or the unit cost method of depreciation

Theoretical Components

Resources:
PDF file: Week 5/6 Notes & Exercises

The following link provides additional information on sequences
https://www.mathsisfun.com/algebra/sequences-series.html
The fascinating world of Fibonacci numbers
https://www.youtube.com/watch?v=iEnR8zupK0A

Arithmetic sequence:
$$T_n = a + (n-1)d$$
Sum of an arithmetic sequence:
$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Order
1. Read through the notes and examples
2. Work through the exercises
3. Complete the investigation at the end of the booklet.
4. Complete the reflection at the end of the booklet
5. Come and see your teacher and make sure you are up to date.

Practical Components

Work through the exercises and show the completed tasks to your teacher.

Be sure to ask for help as you need for the successful completion of all tasks.

Remember to regularly check Google Classroom for messages.

Knowledge Checklist
- Sequences and series
- Functional definition
- Iterative definition
- Arithmetic sequences
- Sum of terms of AS

Investigation

Complete the task at the end of the booklet and submit your work for checking. 😉

Quiz/forum/ other

Remember to check hawkermaths.com **for each week's learning brief.**
Make sure you have joined Google Classroom. If you have not, see your teacher.

MATHEMATICAL APPLICATIONS 3

WEEK 5/6 NOTES & EXERCISES

SEQUENCES AND SERIES

Number patterns

An important skill in mathematics is to be able to:

- Recognise patterns in sets of numbers,
- Describe the patterns in words, and
- Continue the patterns.

Sequences of numbers play an important part in our everyday life. For example, the following sequence: 2.25, 2.34, 2.58, 2.49, 2.65, … gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?

The following sequence is more predictable:

10 000, 9000, 8100, …

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. This compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted: during the second minute, 9000, and so on. Can you predict the next number in the sequence? You are correct if you said 7 290. Each successive term is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They may be finite in number or infinite. Number sequences may follow an easily recognisable pattern, or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting to name a few uses.

Sequences which follow a pattern can be described several different ways. They may be listed in sequential order, they may be described as a functional definition, or they may be described in an iterative definition.

Example: Listing in sequential order

3, 7, 11, 15, … forms a number sequence. The first term is 3, second term is 7, third term is 11, etc. We can describe this pattern in words:
"The sequence starts at 3 and each term is 4 more than the previous one."
Thus, the fifth term is 19, and the sixth term is 23, etc.

EXERCISE 1

1. Write down the first four terms of the sequence if you start with:

a. 4 and add 9 each time

b. 45 and subtract 6 each time

c. 2 and multiply by 3 each time

d. 96 and divide by 2 each time

2. Desribe the following number patterns and write down the next 3 terms:

a. 1, 4, 9, 16, …

b. 1, 8, 27, 64, …

FUNCTIONAL DEFINITION

A functional definition is expressed in the form: $t_n = 2n - 7, n \in \{1,2,3,4,\ldots\}$. Using this definition, the *n*th term can be readily calculated. For this example: $t_1 = 2 \times 1 - 7 = -5$, $t_2 = 2 \times 2 - 7 = -3$, $t_3 = 2 \times 3 - 7 = -1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value n = 100 into the expression for t_n.

Example

Find the first four terms of the sequence: $d_n = 4.9n^2, n \in \{1,2,3,\ldots\}$

Solution

$d_1 = 4.9 \times 1^2 = 4.9$, $d_2 = 4.9 \times 2^2 = 19.6$, $d_3 = 4.9 \times 3^2 = 44.1$ and $d_4 = 4.9 \times 4^2 = 78.4$.

The sequence is $\{4.9,19.6,44.1,78.4,\ldots\}$

EXERCISE 2

Find the first, fifth and tenth terms in the following sequences:

	t_1	t_5	t_{10}
1. $t_n = 2n - 5$			
2. $t_n = \frac{n}{n+1}$			
3. $t_n = (-1)^n + n$			
4. $t_n = n^2 - n + 41$			

ITERATIVE DEFINITION

An iterative definition is expressed in the form: $t_{n+1} = 3t_n - 2, t_1 = 6$. This definition looks complicated, but it is actually straight forward. The word iteration means the calculation of the next term from the previous ter, using the same procedure. The symbol t_{n+1} simply means the next term after the term t_n.

Example

Find the first four terms of the sequence above: $t_{n+1} = 3t_n - 2, t_1 = 6$

Solution

The first term, t_1, is 6 (this is given in the definition).
The second term, t_2, is $3 \times 6 - 2 = 16$
The third term, t_3, is $3 \times 16 - 2 = 46$
The fourth term, t_4, is $3 \times 46 - 2 = 136$

EXERCISE 3

1. Find the first 4 terms in the following sequences.

   a. $t_{n+1} = t_n + 2, t_1 = 3$

   b. $t_{n+1} = 3t_n, t_1 = 2$

   c. $t_{n+1} = t_n - 7, t_1 = 14$

   d. $t_{n+2} = t_{n+1} + t_n, t_1 = 1, t_2 = 1$

ARITHMETIC SEQUENCES

An arithmetic sequence is a sequence where there is a common different between any two successive terms.

For example: 2, 5, 8, 11, 14, … is arithmetic as $5 - 2 = 8 - 5 = 11 - 8 = 14 - 11$, etc, because they each have a difference of 3.
Likewise, 31, 27, 23, 19, … is arithmetic as $31 - 27 = 27 - 23 = 23 - 19$, etc, because they each have a difference of -4.

The rule for finding d is $d = t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots$
This gives the general common difference $d = t_n - t_{n-1}$

Why 'arithmetic'

If a, b, and c are any three consecutive terms of an arithmetic sequence then $b - a = c - b$ (equating common differences). Therefore, through rearranging $2b = a + c$, which gives $b = \frac{a+c}{2}$. The middle term is the arithmetic mean (average) of the terms on each side of it. Hence the name arithmetic sequence.

The General Term Formula

In an arithmetic sequence, the first term is denoted by $\boldsymbol{a}$ and the common difference by $\boldsymbol{d}$. The position of the term is denoted by $\boldsymbol{n}$.

$$\boldsymbol{t_n = a + (n-1)d}$$

For an arithmetic sequence

$t_1 = a$

$t_2 = t_1 + d = a + d$

$t_3 = t_2 + d = (a + d) + d = a + 2d$

$t_4 = t_3 + d = (a + 2d) + d = a + 3d$

$t_5 = t_4 + d = (a + 3d) + d = a + 4d$

Following this pattern, the general term is

$t_n = a + (n-1)d$

Example

Consider the sequence 2, 9, 16, 23, 30,… For this sequence, $a = 2$ and $d = 7$

2	9	16	23
a	$a + d$	$a + d + d$	$a + d + d + d$
a	$a + d$	$a + 2d$	$a + 3d$

Thus, for a given position, n, the term is given by $a + (n-1)d$

For the sequence above, the formula for its general term is given by
$T_n = 2 + (n-1) \times 7$
So, the 50th term is $2 + (50 - 1) \times 7 = 2 + 49 \times 7 = 2 + 343 = 345$

Example

Find k given that 3k+1, k and -3 are consecutive terms of an arithmetic sequence.

Solution

Since the terms are consecutive, $\mathrm{k} - (3\mathrm{k} + 1) = -3 - \mathrm{k}$ (2nd term – 1st term = 3rd term – 2nd term).

$\mathrm{k} - 3\mathrm{k} - 1 = -3 - \mathrm{k}$ (removing of brackets)

$-2\mathrm{k} - 1 = -3 - \mathrm{k}$ (collecting like terms)

$-1 + 3 = 2\mathrm{k} - \mathrm{k}$ (rearranging)

Therefore, $\mathrm{k} = 2$

Example

Find the general term T_n for an arithmetic sequence with $T_3 = 8$ and $T_8 = -17$

Solution

$T_3 = 8$ which tells us, $a + 2d = 8$ … (1) using $T_n = a + (n-1)d$

$T_8 = -17$ which tells us, $a + 7d = -17$ … (2)

We now solve (1) and (2) simultaneously

$a + 2d = 8$

$a + 7d = -17$

(1)-(2) gives $-5d = 25$

Therefore $d = -5$

Substitute $d = -5$ into (1) gives

$a + 2(-5) = 8$

$a - 10 = 8$

$a = 18$

Therefore, the sequence is 18, 13, 8, 3, -2, etc and the general equations is $T_n = 18 + (n-1) \times -5$

EXERCISE 4

1. Consider the sequence 6, 17, 28, 39, 50, …

a. Show that the sequence is arithmetic (Check for a common difference).

b. Find the formula for its general term.

c. Find the 50^{th} term.

d. Is 325 a member of the sequence?

2. Consider the sequence 87, 83, 79, 75, …

a. Show that the sequence is arithmetic.

b. Find the formula for the general term.

c. Find the 40^{th} term.

d. Is -143 a member of the sequence?

3. Show that the following sequences are arithmetic:

a. {-0.12, 3.48, 7.08, …}

b. $\left\{\frac{5}{9}, -\frac{1}{9}, -\frac{7}{9}, \dots\right\}$

c. $\left\{5\frac{2}{3}, 7\frac{4}{15}, 8\frac{13}{15}, \dots\right\}$

d. $\{x + 9, 2x + 7, 3x + 5, \dots\}$

4. For the arithmetic sequence {22, m, n, 37, …}, find the values for m and n.

5. Find k given the consecutive arithmetic terms:

a. 32, k, 3

b. k + 1, 2k + 1, 13

6. Find the general term T_n for an arithmetic sequence given that:

a. $T_7 = 41$ and $T_{13} = 77$

b. The seventh term is 1 and fifteenth term is -39

SUM OF A GIVEN NUMBER OF TERMS OF AN ARITHMETIC SEQUENCE

When the terms of an arithmetic sequence are added together, an arithmetic series is formed.
So, 5, 9, 13, 17, 21, … is an arithmetic sequence,
Whereas 5 + 9 + 13 + 17 + 21 + … is an arithmetic series.

The sum of *n* terms of an arithmetic sequence is given by S_n.
The formula for the sum of *n* terms of an arithmetic sequence when the value of *a* and *d* are known is given by;

$$\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$$

Use $S_n = \frac{n}{2}(a + \ell)$ if we know a and the last term.

Example

Find the sum of 4 + 7 + 10 + 13 + … to 50 terms.

Solution

The series is arithmetic with $a = 4, d = 3$ and $n = 50$.
Using $S_n = \frac{n}{2}[2a + (n-1)d]$ gives

$$S_n = \frac{50}{2}[2 \times 4 + (50-1) \times 3]$$
$$S_n = 25[8 + 49 \times 3]$$
$$S_n = 3875$$

EXERCISE 5

1. Find the sum of the following:

   a. 3 + 7 + 11 + 15 + … to 20 terms

   b. 100 + 93 + 86 + 79 + … to 40 terms

   c. $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + \ldots$ to 50 terms

2. Find the sum of the first 100 positive integers.

3. The first term in an arithmetic sequence is 5 and the sum of the first 20 terms is 1240. Find the common difference, d.

4. The first term is 50 and the 10^{th} term is -40. Find S_{10}

5. Find the sum of 5 + 8 + 11 + 14 + … + 101.
Note: First use $T_n = a + (n-1)d$ to find the value of *n*.

APPLICATIONS OF ARITHMETIC SEQUENCES AND SERIES

EXERCISE 6

1. A diving vessel descends below the surface of the water at a constant rate so that the depth of the vessel after 1 minute, 2 minutes and 3 minutes is 50 meters, 100 meters and 150 meters.

   a. How much is the depth increasing each minute?
   b. What will the depth of the vessel be after 4 minutes?
   c. Continuing at this rate, what will be the depth of the vessel after 10 minutes?
   d. If *n* is the number of minutes it takes to reach a depth of 120 meters, solve for *n*.

2. A worker at a factory is stacking cylindrical-shaped pipes which are stacked in layers. Each layer contains one pipe less than the layer below it. There are 6 pipes in the topmost layer, 7 pipes in the next layer, and so on.

   a. Form an equation for the number of pipes in the *n*th layer.
   b. How many pipes are in the 10^{th} layer?

3. A rare figurine was purchased for $60 and ten years later it is worth $460.

a. The figurine appreciated in value by a constant amount each year. How much did it appreciate each year?
b. What will the value of the figurine be in another 10 years' time?

4. A telecommunications company sells 1600 mobile phones in the first month of its operation. The company plans to increase its sales by 200 mobile phones each month.

a. How many phones does the company plan to sell in the last month of the 5^{th} year of its operation?
b. How many phones does the company plan to sell in the entire 5-year period?
c. How long will it take the company to sell 8600 mobile phones? (use trial and error)

5. Sam starts his career with a monthly wage of $3500. At the beginning of each year that follows he receives a raise and his monthly wage for that year will be $140 greater than the previous year.

a. What will be his yearly salary in the second year of his service?
b. Write down an expression for the total amount earned in *m* years. Hint: which formula would you use, T_n or S_n? Do we need to convert any units?

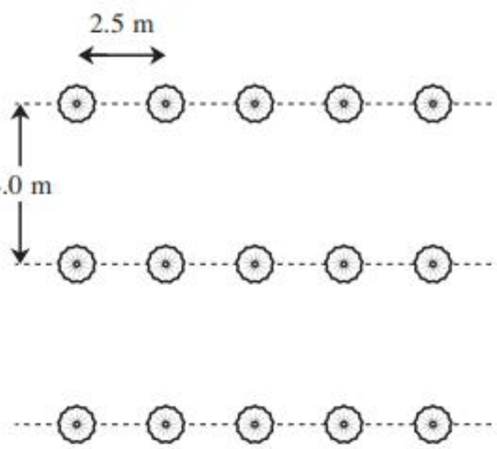
6. Jane is learning to drive. Her first lesson is 24 minutes long, and each subsequent lesson is 2 minutes longer than the lesson before.

a. How long will her 12^{th} lesson be?
b. If Jane reaches 35.4 total *hours* of driving in her *n*th lesson, solve for *n*.

AST STYLE QUESTION

On a beach, umbrellas are placed at precise locations to protect people from the sun. The umbrellas are arranged in straight, equally spaced, parallel rows. Each umbrella is placed in a hole in the sand. As indicated in the figure below, adjacent holes in a row are 2.5 metres apart and adjacent rows are 4.0 metres apart.

People can hire the umbrellas for $10 per half-day, or $15 per day.



Assume that:

- umbrellas are 1.0 metre in diameter;
- umbrellas are precisely upright;
- when viewed from above, the hole is at the centre of the umbrella;
- the length of a row is the distance between the centres of the umbrellas at the extreme ends of the row (where the distance between adjacent holes in a row is 2.5 metres); and
- the width of a set of parallel rows is the distance between the rows that are furthest apart (where the distance between adjacent rows is 4.0 metres).

i) How much would be charged for the hire of a row of beach umbrellas 15 metres long for half a day?

A $60 **B** $65 **C** $70 **D** $75

ii) If there were 10 rows, each 50 metres long, how many umbrella holes would there be?

A 210 **B** 200 **C** 82 **D** 80

iii) Which of the following is a formula for determining how many beach umbrellas can be placed in a single row of a given length, L (where L is a multiple of 2.5)?

A $\frac{L}{2.5-1}$ **B** $\frac{L}{2.5+1}$ **C** $\frac{L}{2.5}+1$ **D** $\frac{L}{2.5}+\frac{L}{2.5-1}$

iv) In one day, what is the most that could be charged for the hire of beach umbrellas in two parallel rows each 12.5 metres long?

A $90 **B** $120 **C** $180 **D** $240

INVESTIGATION WEEK 5/6

Five participants in a half marathon are to receive a share of the total prize pool of $100,000. Three proposals are produced:

Proposal 1: The prizes are to form an arithmetic sequence with the 5^{th} prize being $5,000.

Proposal 2: The prizes are to form an arithmetic sequence with the 1^{st} prize being $40,000.

Proposal 3: The prizes are to form an arithmetic sequence with the 3^{rd} and 4^{th} prizes being $20,000 and $15,000 respectively.

Complete the table below to investigate which proposal is better for the:

a) winner of the marathon

b) 5^{th} place getter in the marathon.

Place	**PROPOSAL**		
	1	**2**	**3**
1^{st}			
2^{nd}			
3^{rd}			
4^{th}			
5^{th}			
Total			

Show necessary working here.

MARKING RUBRIC

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation	Student completes the investigation of the brief to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and communications	Student responses are accurate and appropriate in presentation of mathematical ideas in different contexts, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and techniques	Student submitted work selects and applies appropriate mathematical modelling and problem solving techniques to solve practical problems, and demonstrates proficiency in the use of mathematical facts, techniques and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
		FINAL			**/20**

Student Reflection:
How did you go with this week's work?

What was interesting?

What did you find easy?

What do you need to work on?