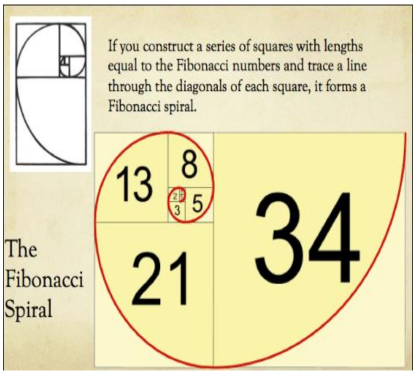
Week 12/13
Term 2
2024

HAWKER COLLEGE
Engage | Inspire | Achieve

MA3
Growth and decay in sequences

Goals



This fortnight we are going to:

- use a general first-order linear recurrence relation to generate the terms of a sequence and to display it in both tabular and graphical form
- recognise that a sequence generated by a first-order linear recurrence relation can have a long term increasing, decreasing or steady-state solution
- use first-order linear recurrence relations to model and analyse (numerically or graphically only) practical problems; for example, investigating the growth of a trout population in a lake recorded at the end of each year and where limited recreational fishing is permitted, or the amount owing on a reducing balance loan after each payment is made

Theoretical Components

Resources:
PDF file: Week 12/13 Notes & Exercises

The fascinating world of Fibonacci numbers
https://www.youtube.com/watch?v=iEnR8zupK0A

Order
1. Read through the notes and examples
2. Work through the exercises
3. Complete the investigation at the end of the booklet.
4. Complete the reflection at the end of the booklet
5. Come and see your teacher and make sure you are up to date.

Practical Components

Work through the exercises and show the completed tasks to your teacher.

Be sure to ask for help as you need for the successful completion of all tasks.

Remember to regularly check Google Classroom for messages.

Knowledge Checklist
- Generating sequences using recurrence relations
- Finding a recurrence relation for a given sequence
- Applications of recurrence relations
- Using spreadsheets to model recurrence relations

Investigation

Complete the task at the end of the booklet and submit your work for checking. ☺

Quiz/forum/ other

Remember to check hawkermaths.com **for each week's learning brief.**
Make sure you have joined Google Classroom. If you have not, see your teacher.

WEEK 12/13 NOTES & EXERCISES

A recurrence relation, also called a difference equation, is a rule that specifies a particular term in a sequence using the previous term or terms.

Consider the sequence, 3, 8, 13, 18, …

The first term in the sequence is 3 and each subsequent term in the sequence is 5 more than the previous term.

We can write this as $t_{n+1} = t_n + 5, t_1 = 3$

This is called a first order difference equation because it links consecutive terms in the sequence.

In the difference equation $t_{n+1} = t_n + 5,$ where $t_1 = 3$, t_1 represents the first term in the sequence, t_n represents a particular term in the sequence and t_{n+1} represents the term in the sequence that comes immediately after t_n.

The same recurrence relation could also be written as $t_n = t_{n-1} + 5$ where $t_1 = 3$, t_1 represents the first term in the sequence, t_{n-1} represents a particular term in the sequence and t_n represents the term in the sequence that comes immediately after t_{n-1}.

The first term in a sequence is usually denoted by t_1.

GENERATING SEQUENCES USING RECURRENCE RELATIONS

An iterative definition is expressed in the form: $t_{n+1} = 3t_n - 2, t_1 = 6$. This definition looks complicated, but it is actually straight forward. The word iteration means the calculation of the next term from the previous term, using the same procedure. The symbol t_{n+1} simply means the next term after the term t_n.

Example

Find the first four terms of the sequence above: $t_{n+1} = 3t_n - 2, t_1 = 6$

Solution

The first term, t_1, is 6 (this is given in the definition).
The second term, t_2, is $3 \times 6 - 2 = 16$
The third term, t_3, is $3 \times 16 - 2 = 46$
The fourth term, t_4, is $3 \times 46 - 2 = 136$

EXERCISE 1

1. Find the first 4 terms in the following sequences.

   a. $t_{n+1} = t_n + 2, t_1 = 3$

   b. $t_{n+1} = 3t_n, t_1 = 2$

   c. $t_{n+1} = t_n - 7, t_1 = 14$

   d. $t_n = 2t_{n-1} + 1, t_1 = 3$

2. Write down the difference equation of the sequence if you start with:

   a. 4 and add 9 each time

   b. 45 and subtract 6 each time

   c. 2 and multiply by 3 each time

   d. 96 and divide by 2 each time

3. The difference equation $u_n = 2u_{n-1} + 5$ can be used to generate a sequence. If $u_4 = 9$, find u_6.

4. The third term in a sequence is 172. If a recurrence relation that generates the sequence is $t_{n+1} = 3t_n - 2$, find the first term in the sequence.

FINDING A RECURRENCE RELATION FOR A GIVEN SEQUENCE

Sometimes it is necessary to determine a difference equation or a recurrence relation for a given sequence. In some cases, this can be done by inspection. In other cases, algebraic techniques need to be used.

The rule for a first order reccruence relation takes the form

$\boldsymbol{t_{n+1} = rt_n + d}$ with $t_1 = a$ (first term) *or* $\boldsymbol{t_n = rt_{n-1} + d}$

Note: For an arithmetic sequence $r = 1$ and for a geometric sequence $d = 0$

The sequences we will be dealing with here may be neither arithmetic or geometric.

Example

The sequence 8, 26, 80, … is generated from a first order recurrence relation. Find this recurrence relation.

Solution

Write down the general rule for a first order recurrence relation.	$t_{n+1} = rt_n + d$ with $t_1 = a$
Write each term in the sequence with its term number.	$t_1 = 8,\ t_2 = 26,\ t_3 = 80$
Since $t_1 = a$, state the value of a.	$t_1 = 8$ $a = 8$
The recurrence relation $t_{n+1} = rt_n + d$ has two unknowns, d and r. Use the three terms to write down two equations in terms of d and r.	$t_2 = rt_1 + d$ $26 = 8r + d$ [1] $t_3 = rt_2 + d$ $80 = 26r + d$ [2]
Solve simultaneously to find the values of d and r.	[2] − [1]: $54 = 18r$ $\frac{54}{18} = r$ $r = 3$ Substitute $r = 3$ into [1]. $26 = 8 \times 3 + d$ $= 24 + d$ $d = 2$
Write the recurrence relation.	$t_{n+1} = 3t_n + 2$ with $t_1 = 8$

EXERCISE 2

1. Find recurrence relations that generate each of the following sequences.

a. 4, 8, 12, 16, … $t_n =$ ... , $t_1 =$ ...

b. 3, 6, 12, 24, …

c. 100, 50, 25, 12.5, …

d. 49, 42, 35, 28, …

2. The recurrence relation $t_n = 2t_{n-1} + d$ where $t_1 = 12$, defines the sequence 12, -6, -42, …

a. Find the value of d.

b. Find the seventh term of the sequence.

3. The first three terms of the sequence defined by the recurrence relation $t_{n+1} = rt_n + d$ with $t_1 = a$ are 28, 50, and 94. Find the values of r, d and a.

APPLICATIONS OF RECURRENCE RELATIONS

Recurrence relations can be used to model many real-life situations, including compound interest, loans and investments, depreciation, drug doses and population growth and decay.

When writing recurrence relations, it is important to identify or calculate the value of the first term and to determine the relationship between successive terms in the sequence.

Example

A business purchased a photocopier for $10 000 in 2014. It depreciated at a rate of 18% per annum. Write down a recurrence relation that gives the value of the photocopier at the start of each year,

Solution

If the photocopier depreciates by 18% per annum, this means that it loses 18% of its value from one year to the next. Thus, the value of the photocopier in any year is 82% of its value the previous year.

The recurrence relation can be written as

$V_{n+1} = 0.82V_n$ (or $V_n = 0.82V_{n-1}$) where $V_1 = 10\ 000$

EXERCISE 3

1. Write a recurrence relation to represent each of the following situations.

a. Elly receives an annual salary of $40 000. At the start of each year, she receives a 4% pay rise.

b. A frog jumps 48 cm in its first jump and the length of each subsequent jump is 80% of its previous jump length.

c. Carla parks her car in a car park. She is charged $12 for the first hour and an additional $8 for each subsequent hour.

d. A rain water tank contains 5000 litres. Each day 12% of the volume of water in the tank is used and in the evening an extra 200 litres of water is added to the tank.

2. Park authorities need to control the size of a kangaroo population in a national park. Each year there is a 15% increase in the size of the population due to breeding and at the end of the year 300 kangaroos are relocated out of the park. The size of the kangaroo population at the start of 2022 was 2000. What is the number of kangaroos living in the park at the start of 2024?

3. A mobile phone depreciates in value by a constant amount per month and its value is given by the explicit rule:

$V_n = 1200 - 20n$, where V_n is the balance (value in dollars) after *n* months

a. How much does the value of the phone depreciate each month?

b. What is the purchase price of the phone?

c. Write a recursive rule for V_n in terms of V_{n-1}, and an initial condition V_0.

4. A council employs contract gardeners to plant and maintain public gardens in a new housing estate. In the first month, the gardeners work for 800 hours, in the second month, they work for 680 hours and in the third month, they work for 584 hours. This decreasing patter of work continues.

a. Show that the sequence of hours works is neither arithmetic or geometric.

b. Write a recurrence relation that models the number of hours the gardeners will work in the nth month.

c. How many hours will the gardeners work in the sixth month? Answer correct to one decimal place.

d. How many more hours will the gardeners work for in the sixth month than the seventh month?

USING SPREADSHEETS TO MODEL RECURRENCE RELATIONS

Example

Use a spreadsheet to generate 20 values for the sequence, arising from the difference equation
$t_{n+1} = 0.8t_n + 1.2$; $t_1 = 0.2$.

The completed spreadsheet looks like this. Note that you only fill in Row 4 and then use the Fill Down feature.

This is neither an arithmetic or geometric sequence and there is no common difference and no common ratio.

	A	B
1	Worked Example 14	
2	n	t_n
3	1	0.2
4	=A3+1	=0.8*B3+1.2
5	=A4+1	=0.8*B4+1.2
6	=A5+1	=0.8*B5+1.2
7	=A6+1	=0.8*B6+1.2
8	=A7+1	=0.8*B7+1.2
9	=A8+1	=0.8*B8+1.2
10	=A9+1	=0.8*B9+1.2
11	=A10+1	=0.8*B10+1.2
12	=A11+1	=0.8*B11+1.2
13	=A12+1	=0.8*B12+1.2
14	=A13+1	=0.8*B13+1.2
15	=A14+1	=0.8*B14+1.2
16	=A15+1	=0.8*B15+1.2
17	=A16+1	=0.8*B16+1.2
18	=A17+1	=0.8*B17+1.2
19	=A18+1	=0.8*B18+1.2
20	=A19+1	=0.8*B19+1.2
21	=A20+1	=0.8*B20+1.2
22	=A21+1	=0.8*B21+1.2

	A	B
1	Worked Example 14	
2	n	t_n
3	1	0.20
4	2	1.36
5	3	2.29
6	4	3.03
7	5	3.62
8	6	4.10
9	7	4.48
10	8	4.78
11	9	5.03
12	10	5.22
13	11	5.38
14	12	5.50
15	13	5.60
16	14	5.68
17	15	5.74
18	16	5.80
19	17	5.84
20	18	5.87
21	19	5.90
22	20	5.92

Example

Steve obtains a loan of $2,000 from a bank, with monthly interest of 0.50% (that is, 6% per annum) and monthly repayments of $60.

Using the difference equation $t_{n+1} = rt_n + d$ with an initial term $2000 and d being -60 (as he is reducing the loan)

$r = 1 + \frac{0.5}{100}$ which becomes 1.005
Note that we start the first month as 0 and again we only need to complete Row 3 and the Fill Down.
There are missing rows which were simply removed to save space. You do not need to do this.
The 37th Row shows -$26.44 which means that he only had to make a payment of $60 - $26.64 = $33.36 in the 37th month.
In summary, Steve repaid 36 × 60 + 33.64 = $2193.36 over the 37 months, which means he paid $193.36 in interest.

	A	B
1	Worked Example 15	
2	Month	Debt
3	0	2000
4	=A3+1	=1.005*B3-60
5	=A4+1	=1.005*B4-60
6	=A5+1	=1.005*B5-60
7	=A6+1	=1.005*B6-60
8	=A7+1	=1.005*B7-60
9	=A8+1	=1.005*B8-60
33	=A32+1	=1.005*B32-60
34	=A33+1	=1.005*B33-60
35	=A34+1	=1.005*B34-60
36	=A35+1	=1.005*B35-60
37	=A36+1	=1.005*B36-60
38	=A37+1	=1.005*B37-60
39	=A38+1	=1.005*B38-60
40	=A39+1	=1.005*B39-60

	A	B
1	Worked Example 15	
2	Month	Debt
3	0	$2,000.00
4	1	$1,950.00
5	2	$1,899.75
6	3	$1,849.25
7	4	$1,798.49
8	5	$1,747.49
9	6	$1,696.22
33	30	$386.00
34	31	$327.93
35	32	$269.57
36	33	$210.92
37	34	$151.97
38	35	$92.73
39	36	$33.19
40	37	-$26.64

EXERCISE 4

1. A couple wishes to buy a house and plans to borrow $100,000 with monthly repayments of $800 and a monthly interest of 0.50%.

    a. Construct a difference equation which may be used to calculate the size of the debt at the end of each month.

    b. Construct a spreadsheet which calculates the size of the debt each month.

    c. From your spreadsheet, find how many months are required to fully repay the debt and how much is repaid in the last payment.

    d. Calculate how much it has cost the couple in total for the loan of the $100,000.

2. A vegetable farmer uses water from a storage tank to water his crops. The tank initially contains 50,000 litres. Each day, 10% of the water in the tank is used to water the crops and each evening an extra 3,000 litres is pumped into the tank from a nearby river. This pattern continues each day.

    a. Write a difference equation for this scenario.

    b. Use a spreadsheet to determine on which morning the volume of water first falls below 40,000 litres.

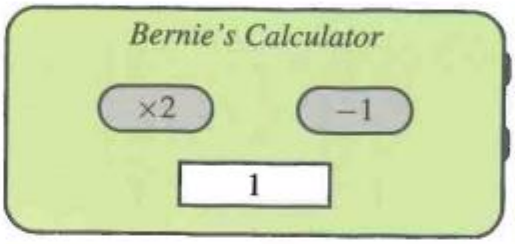
    c. How many litres of water will be contained in the tank in the long term?

AST STYLE QUESTION

Bernie has bought a cheap calculator. It has only two operation buttons:

$\times 2$ doubles the number shown

-1 subtracts 1 from the number shown



For instance, with the calculator starting at 1, to get to 3 she would press $\times 2$ $\times 2$ -1.

Bernie likes to start at 1 and use as few button presses as she can to get to any particular number.

The calculator starts at 1. For each of the given numbers, what is the fewest number of button presses she needs to reach it?

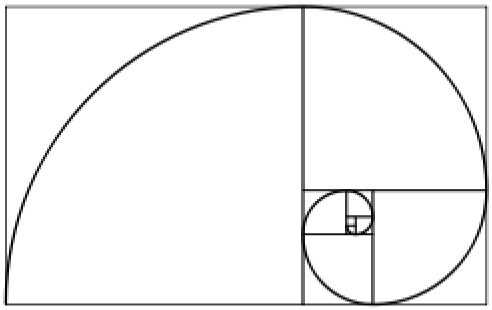
A. 12

B. 27

C. 72

INVESTIGATION WEEK 12/13

The Fibonacci and Lucas Sequences



Leonardo Fibonacci of Pisa was a mathematician in the 12th century, Italy. He discovered a number series from which one can derive the Golden Mean by charting the population of rabbits. Here is the beginning of the sequence:

1, 1, 2, 3, 5, 8,13, 21, 34, …

The Lucas numbers or Lucas series are an integer sequence named after the mathematician François Édouard Anatole Lucas (1842-1891). The sequence named after him is closely related to the Fibonacci sequence.

These sequences are defined recursively by:

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}, \text{for } n \geq 3$$

$$L_1 = 1, L_2 = 3, L_n = L_{n-1} + L_{n-2}, \text{for } n \geq 3$$

Each number is the sum of the two proceeding numbers.

1. Write out the first 12 terms of each sequence.

2. Explain why every third term of each sequence is even and the rest are odd.

3. Write out the sequence $L_1 + F_1, L_2 + F_2, L_3 + F_3, \ldots$ and $L_1 - F_1, L_2 - F_2, L_3 - F_3, \ldots$

4. How do the two sequences relate to the Fibonacci sequence?

MARKING RUBRIC

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation	Student completes the investigation of the brief to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and communications	Student responses are accurate and appropriate in presentation of mathematical ideas in different contexts, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and techniques	Student submitted work selects and applies appropriate mathematical modelling and problem solving techniques to solve practical problems, and demonstrates proficiency in the use of mathematical facts, techniques and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
		FINAL			**/20**

Student Reflection:
How did you go with this week's work?

What was interesting?

What did you find easy?

What do you need to work on?