Week 4/5
Term 3
2023

HAWKER COLLEGE
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Learning Brief

MA4

Table 1. Even Principal Payment Schedule
($10,000 loan, 7% annual interest, 20 annual payments)

Year	Total Payment	Principal	Interest [1]	Unpaid Balance
0				$ 10,000
1	$ 1,200	$ 500	$ 700	$ 9,500
2	$ 1,165	$ 500	$ 665	$ 9,000
3	$ 1,130	$ 500	$ 630	$ 8,500
4	$ 1,095	$ 500	$ 595	$ 8,000
5	$ 1,060	$ 500	$ 560	$ 7,500
6	$ 1,025	$ 500	$ 525	$ 7,000
7	$ 990	$ 500	$ 490	$ 6,500
8	$ 955	$ 500	$ 455	$ 6,000
9	$ 920	$ 500	$ 420	$ 5,500
10	$ 885	$ 500	$ 385	$ 5,000
11	$ 850	$ 500	$ 350	$ 4,500
12	$ 815	$ 500	$ 315	$ 4,000
13	$ 780	$ 500	$ 280	$ 3,500
14	$ 745	$ 500	$ 245	$ 3,000
15	$ 710	$ 500	$ 210	$ 2,500
16	$ 675	$ 500	$ 175	$ 2,000
17	$ 640	$ 500	$ 140	$ 1,500
18	$ 605	$ 500	$ 105	$ 1,000
19	$ 570	$ 500	$ 70	$ 500
20	$ 535	$ 500	$ 35	$ 0
Total	$ 17,350	$ 10,000	$ 7,350	

[1] interest = unpaid balance times 7 percent.

Goals

This brief:

- use a recurrence relation to model a compound interest loan or investment, and investigate (numerically or graphically) the effect of the interest rate and the number of compounding periods on the future value of the loan or investment
- use a recurrence relation to model a reducing balance loan and investigate (numerically or graphically) the effect of the interest rate and repayment amount on the time taken to repay the loan
- with the aid of a financial calculator or computer-based financial software, solve problems involving reducing balance loans; for example, determining the monthly repayments required to pay off a housing loan
- use a recurrence relation to model an annuity, and investigate (numerically or graphically) the effect of the amount invested, the interest rate, and the payment amount on the duration of the annuity
- with the aid of a financial calculator or computer-based financial software, solve problems involving annuities (including perpetuities as a special case); for example, determining the amount to be invested in an annuity to provide a regular monthly income of a certain amount

Theoretical Components

Knowledge Checklist

- Concept of compounding
- Reducing balance
- Term of investment or loan
- Interest rate per period
- Repayment schedule
- Balance after repayment
- Discharge of loan
- Annuity
- Principal
- Calculating growth factor
- Amount owing
- Debited
- Interest period

Practical Components

Exercise 1: Compound Interest
Exercise 2: Reducing Balance Loans
Exercise 3: Annuities
Exercise 4: Spreadsheets

Investigation

Loan Schedules and Annuities

Compound Interest

Consider the case where a bank pays compound interest of 5% per annum on an amount of $20,000. The amount is invested for 4 years, and interest is calculated yearly. Compound interest receives its name because the interest which is earned is paid back into the account so that the next time interest is calculated, it is calculated on an increased amount. There is a compounding effect on the money in the account. If we calculate the amount in the account mentioned above, we will have the following amounts.

Start:	$20,000
After 1 year:	$20,000 x 1.05 = $21,000
After 2 years:	$20,000 x 1.05 x 1.05 = $22,050
After 3 years:	$20,000 x 1.05 x 1.05 x 1.05 = $23,152.50
After 4 years:	$20,000 x 1.05 x 1.05 x 1.05 x 1.05 = $24,310.13

The following formula is used to calculate compound interest:

$$A = PR^n = P\left(1 + \frac{r}{100}\right)^n$$

where:
$\mathrm{A} = \text{the amount at the end of } n \text{ compounding periods, \$}$
$P = \text{principal, \$}$
$r = \text{rate of interest per period}$
$n = \text{number of compounding periods}$

Example 1

Helen inherits $60,000 and invests it for 3 years in an account which pays compound interest of 8% per annum compounding every 6 months.

a. What will be the amount in Helen's account at the end of 3 years?
b. How much will Helen receive in interest over the 3-year period?

Solution

What will be the amount in Helen's account at the end of 3 years?

Step 1: This is an example of compound interest. Use $A = P\left(1 + \frac{r}{100}\right)^n$. Interest is calculated every 6 months over 3 years, this means there are 6 periods, $n = 6$. Interest is 8% per annum or 4% per 6 months, $r = 4$.

$P = 60000$
$n = 6$
$r = 4$

$A = P\left(1 + \frac{r}{100}\right)^n$
$A = 60000\left(1 + \frac{4}{100}\right)^6$
$A = 75919.14$

Step 2: Write your answer.

At the end of 3 years, Helen will have a total amount of $75,919.14

How much will Helen receive in interest over the 3-year period?

Step 1: Interest equals the amount in the account at the end of 3 years, less the amount in the account at the start of the investment

$\text{Interest} = \text{Total amount} - \text{Principal}$
$I = 75919.14 - 60000$
$I = 15919.14$

Step 2: Write your answer.

Amount of interest earned over 3 years is $15,919.14

Example 2

Jim invests $16,000 in a bank account which earns compound interest at the rate of 12% per annum compounding every quarter. At the end of the investment, there is $25,616.52 in the account. For how many years did Jim have his money invested?

Solution

Step 1: We know the value of A, P, and r. We need to find n using the compound interest formula.	$A = 25616.52$ $P = 16000$ $r = \frac{12}{4} = 3$ $25616.52 = 16000\left(1 + \frac{3}{100}\right)^n$ $1.601 = 1.03^n$
Step 2: Trial and error. Try different values of n.	Let n = 5 $\quad 1.03^5 = 1.159$ Let n = 15 $\quad 1.03^{15} = 1.344$ Let n = 16 $\quad 1.03^{16} = 1.605$
Step 3: Write your answer.	It will take 16 periods where a period is 3 months. So, it will take 48 months or 4 years.

Exercise 1

1. $13,000 is invested in an account which earns compound interest of 8%, compounding quarterly.
   a. After 5 years, how much is in the account?

   b. How much interest was earned in that period?

2. $10,000 is invested in an account which earns compound interest of 10% per annum, find the amount in the account after 5 years if the interest is compounding monthly.

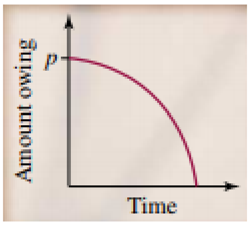
3. In an account earning compound interest of 8% per annum compounding quarterly, an amount of $6,000 is invested. When the account is closed, there is $7,609.45 in the account. How many years was the account open?

Reducing Balance Loans

When we invest money with a financial institution, the institution pays us interest because it is our money they lend to others, as we saw before. Conversely, when we borrow money from an institution, we are using the institution's money and so they charge us interest.

In reducing balance loans, interest is usually charged every month by the financial institution and repayments are made by the borrower on a regular basis. These repayments nearly always amount to more than the interest for the same period of time and so the amount still owing is reduced. Since the amount still owing is continually decreasing and interest is calculated on a daily balance but debited monthly, the amount of interest charged decrease as well throughout the life of the loan.

This means that less of the amount borrowed is paid off in the early stages of the loan compared to the end. That is, the rate at which the loan is paid off increase as the loan progresses.



The terms below are often used when talking about reducing balance loans:

- Principal, P = amount borrowed ($)
- Balance, A = amount still owing ($)
- Term = life of the loan (years)
- To discharge a loan = to pay off a loan (that is A = $0)

It is possible to have an 'interest only' loan account where the repayments equal the interest added and so the balance doesn't reduce. This option is available to a borrower who wants to make the smallest repayment possible.

Loan Schedules

The first amount of interest is added to the balance of a loan account one month after the funds are provided to the customer, the first repayment is usually made on the same day. Consider a loan of $800 that is repaid in 5 monthly instalments of $165.81 at an interest rate of 1.2% per month, interest debited each month. A loan schedule can be drawn from this information, showing all interest debits and repayments. From the schedule the amount owing after each month is shown and the total interest charged can be calculated. For any period of the loan:

$$\text{Total repayments} = \text{Interest paid} + \text{Principal repaid}$$

Month	Balance at start of month ($)	Interest (1.2% of monthly starting balance) ($)	Total owing at the end of month ($)	Repayment ($)	Balance after repayment ($)
1	800.00	800.00 x 0.012 = 9.6	800.00 + 9.6 = 809.60	165.81	809.60 – 165.81 = 643.79
2	643.79	7.73	651.52	165.81	485.71
3	485.71	5.83	491.54	165.81	325.73
4	325.73	3.91	329.64	165.81	163.83
5	163.83	1.97	165.80	165.80	0.00

Each month interest of 1.2% of the monthly starting balance is added to that balance and the repayment value is subtracted, leaving the starting balance for the next month. This process continues until the loan is paid off after 5 months. **Note:** the amount of interest charged falls each month and so the amount of principal paid each month increases as outlined earlier.

Balance at the start of the second month = balance at the start of the first month x R – repayment ($A_2 = A_1 \times R - Q$), where $R = 1 + \frac{r}{100}$.

As mentioned earlier, institutions usually debit a loan account with interest each month. We can also consider situations in which interest is debited fortnightly and quarterly. The frequency with which a customer can make repayments may be weekly, fortnightly, monthly or quarterly. In all cases in this booklet, the frequency of debiting interest will be the same as the frequency of making repayments, although this is not necessary in practice, but it will make calculations easier.

The calculations outlined for monthly repayments will be followed for all other repayment frequencies.

Example

A loan of $16,000 is repaid by monthly instalments of $430.83 over 4 years at an interest rate of 1.1% per month, interest debited monthly. Calculate:

a. The amount still owing after the 5th repayment
b. The decrease in the principal during the first 5 repayments
c. The interest charged during this time

Solution

The amount still owing after the 5th repayment

Step 1: Calculate the growth factor, R.

$$R = 1 + \frac{r}{100}$$
$$R = 1 + \frac{1.1}{100} = 1.011$$

Step 2: Find the balance, A_1, at the end of the 1st month (or the start of the 2nd month).
$A_0 = 16000, Q = 430.83$

$$A_1 = A_0 \times R - Q$$
$$A_1 = 16000 \times 1.011 - 430.83$$
$$A_1 = \$15745.1$$

Step 3: Find A_2 from A_1, repeat until A_5 is found.
A_5 is the balance at the end of the 5th month.

$$A_2 = A_1 \times R - Q$$
$$A_2 = 15487.54$$
$$A_3 = 15227.07$$
$$A_4 = 14963.74$$
$$A_5 = 14697.51$$

Step 4: Write your answer.

The amount owing after 5 months is $14,697.51.

The decrease in the principal during the first 5 repayments

Step 1: The decrease in the principal is the difference between the amount owing initially, A_0, and after the 5th month, A_5.	Decrease in principal $A_0 - A_5$ $= 16000 - 14697.51$ $= 1302.49$
Step 2: Write your answer.	The principal has decreased by \$1302.49 in the first 5 months of the loan.

The interest charged during this time

Step 1: Interest charged = Total repayments – Principal repaid	Interest charged $430.83 \times 5 - 1302.49$ $= 851.66$
Step 2: Write your answer.	The interest charged during the first 5 months is \$851.66

Exercise 2

1. A loan of \$1000 is repaid in five monthly instalments of \$206.04 at a rate of 1% per month, interest debited monthly. Calculate:
    a. The amount still owing after the 4th repayment.
    b. The total interest charged during the 5 months.

2. A loan of \$2000 is repaid in four quarterly instalments of \$525.25 at a rate of 2% per quarter, interest debited quarterly. Calculate:
    a. The amount still owing after the 3rd repayments.
    b. The total interest charged during the 4 quarters.

The annuities formula

Last week step-by-step calculations were made to determine the amount still owing. The process was restrictive in that the previous balance was needed to calculate subsequent balances. A method is needed to enable calculation of the amount still owing at any point in time during the term of the loan.

An annuities formula can be used to enable such calculations to be made. An annuity is a regular payment. When a consumer borrows money from a financial institution that person contracts to make regular payments of annuities in order to repay the sum borrowed over time.

The amount owing in a loan account for *n* repayments is given by the annuities' formula:

$$A = PR^n - \frac{Q(R^n - 1)}{R - 1}, \qquad R = 1 + \frac{r}{100}$$

where:

$\text{A} = \text{amount owing after } n \text{ repayments, } \$$
$P = \text{principal, } \$$
$R = \text{growth factor for amount borrowed}$
$n = \text{number of repayments}$
$\text{Q} = \text{amount of regular repayments made per period}$

Note: Part of the formula $\frac{Q(R^n-1)}{R-1}$ is the same as the formula for the sum of a geometric sequence which follows as loans and investments are geometric progressions.

Example 1

A loan of $50,000 is taken out over 20 years at a rate of 6% p.a. (interest debited monthly) and is to be repaid with monthly instalments of $358.22. find the amount still owing after 10 years.

Solution

Step 1: State the loan amount, *P*, and regular repayments, *Q*.	$P = 50000$ $Q = 358.22$
Step 2: Find the number of payments, *n*, interest rate per month, *r*, and growth factor, *R*.	$n = 10 \times 12 = 120$ $r = \frac{6}{12} = 0.5$ $R = 1 + \frac{0.5}{100} = 1.005$
Step 3: Substitute into the annuities formula and evaluate A.	$A = PR^n - \frac{Q(R^n - 1)}{R - 1}$ $A = 50000 \times 1.005^{120} - \frac{358.22(1.005^{120} - 1)}{1.005 - 1}$ $A = \$32264.98$
Step 4: Write your answer.	The amount still owing after 10 years will be $32,264.98

Note: Even though 10 years is the halfway point of the term of the loan, more than half of the original $50,000 is still owing. When we consider borrowing money, we usually know how much is needed and we choose a term which requires a repayment we can afford. To find the repayment value, Q, the following formula is used.

$$Q = \frac{PR^n(R-1)}{R^n - 1}$$

Example 2

Rob wants to borrow $2,800 for a new hi-fi system from a building society at 7.5% p.a., interest adjusted monthly.

a. What would be Rob's monthly repayment if the loan is fully repaid in 1 year?
b. What would be the total interest charged?

Solution

What would be Rob's monthly repayment if the loan is fully repaid in 1 year?

Step 1: Find *P*, *n*, *r* and *R*.

$P = 2800$
$n = 12$
$r = \frac{7.5}{12} = 0.625$
$R = 1 + \frac{0.625}{100} = 1.00625$

Step 2: Substitute into the annuities formula to find the regular monthly repayments, Q.

$Q = \frac{PR^n(R-1)}{R^n - 1}$
$Q = \frac{2800 \times 1.00625^{12} \times (1.00625 - 1)}{1.00625^{12} - 1}$
$Q = 242.92$

Step 3: Write your answer.

The monthly regular payments are $242.92 over 12 months.

What would be the total interest charged?

Step 1: Total interest = Total repayments – Amount borrowed

$\text{Total Interest} = 242.92 \times 12 - 2800$
$\text{Total Interest} = 2915.04 - 2800$
$\text{Total Interest} = 115.04$

Step 2: Write your answer.

The total interest on a $2,800 loan over 18 months is $115.04

In general, with each repayment more of the loan is paid off and less interest is paid.

Exercise 3

1. A loan of $65,000 is taken out over 20 years at a rate of 12% p.a. (interest debited monthly) and is to be repaid with monthly instalments of $715.71. Find the amount still owing after 5 years.

2. A loan of $52,000 is taken out over 15 years at a rate of 13% p.a. (interest debited fortnightly) and is to be repaid with fortnightly instalments of $303.37. Find the amount still owing after 3 years.

3. Riley borrows $48,000, taken out over 10 years and to be repaid in monthly instalments. (**Note:** As interest rate increases, the monthly repayment also increases if the loan period is to remain the same.) Find the amount still owing after 5 years if interest is debited monthly at a rate of:
   a. 6% p.a. and the repayment of $532.90
   b. 12% p.a. and the repayment of $688.66

4. Gwen has borrowed $14,000 for renovations to her house. The terms of this loan are monthly instalments of $297.46 over 5 years with interest debited monthly at 10% p.a. of the outstanding balance.
   Which of the following options is the correct amount still owing after 3 years:

   a. $A = 14000 \times 1.008333^{36} - \frac{297.46\left(1.008333^{36}-1\right)}{1.008333-1}$

   b. $A = 14000 \times 1.008333^{60} - \frac{297.46\left(1.008333^{60}-1\right)}{1.008333-1}$

   c. $A = 14000 \times 1.1^{60} - \frac{297.46\left(1.1^{60}-1\right)}{1.1-1}$

   d. $A = 14000 \times 1.08333^{36} - \frac{297.46\left(1.08333^{36}-1\right)}{0.008333-1}$

   e. $A = 14000 \times 1.08333^{36} - \frac{297.46\left(1.08333^{36}-1\right)}{1.08333-1}$

5. Ben took out a loan for $20,000 to buy a new car. The contract required he repay the loan over 5 years with monthly instalments of $421.02. After 2 years, Ben used the annuities formula to obtain the expression below to calculate the amount he still owed.

$$A = 20000 \times 1.008^{24} - \frac{421.02(1.008^{24} - 1)}{1.008 - 1}$$

   What is the interest rate per annum charged for this reducing balance loan?

   a. 1.008%

   b. 0.008%

   c. 0.096%

   d. 9.6%

   e. 12.096%

Using a Spreadsheet

On Google Classroom, there is a file called ‘Annuities and Repayments’ which is a formatted spreadsheet to help you with calculating annuities and monthly repayments. Use this spreadsheet for the following exercises.

Exercise 4

1. A loan of $20,000 has interest charged monthly at a rate of 9% p.a. Calculate the amount still owing after 3 years if the term of the loan is:

   a. 4 years and monthly repayments of $497.70 are made?

   b. 8 years and monthly repayments of $293 are made?

2. Amber's loan of $85,000 is charged interest at 7% p.a., interest adjusted monthly. Calculate **(i)** the monthly repayment and **(ii)** total interest charged if the loan is repaid in:
   a. 5 years
   b. 10 years
   c. What do you notice about monthly repayments and amount of interest charged with the different loan terms?

3. Three years ago, Steph borrowed $18,000. The reducing balance loan was for a term of 5 years and was to be repaid in monthly instalments of 10.2% p.a. (adjusted monthly). How much does Steph still owe?

4. Tim has borrowed $450,000 to buy an apartment. He agrees to repay the reducing balance loan over 15 years with monthly instalments at 9.3% p.a. Calculate the monthly instalment and amount still owing after:
   a. 20^{th} repayment
   b. 150^{th} repayment

Ginny takes out a loan of $85,000 to set up a business. She starts with quarterly repayments of $2,300.42 and the loan is due to run for 20 years at 9% p.a., charged quarterly. However, after 1 year the interest rate falls to 8% p.a. and consequently the quarterly repayments fall to $2,143.88 to maintain the 20-year term.

1. What amount is still owing after 2 years?
2. What amount would have still been owing after 2 years if the rate had remained at 9% p.a.?
3. What would be the difference in total interest charged between the two scenarios?

MARKING RUBRIC **Week 4/5** **Name:**

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work, including exercises and any Mathspace and/or other tasks, of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation Task	Student completes the investigation task of the week to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Communication and Reasoning	Student responses are accurate and appropriate in presentation of mathematical ideas in different contexts, with clear and logical working out shown.	**4**	**-**		**/4**
Knowledge and Application	Student submitted work selects and applies appropriate mathematical modelling and problem-solving techniques to solve practical problems and demonstrates proficiency in the use of mathematical facts, techniques, and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the practical work, including exercises and any Mathspace and/or other tasks, and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
				FINAL	**/20**

Student Reflection: How did you go with this week's work? What was interesting? What did you find easy? What do you need to work on?