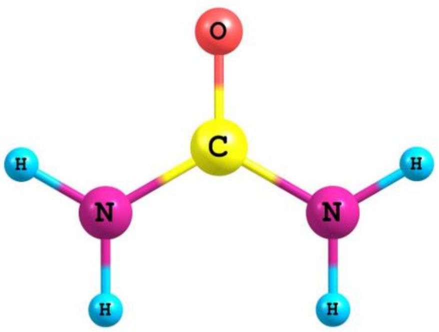
NAME________________

Week 14&15
Term 2
2022

HAWKER COLLEGE
Engage | Inspire | Achieve

Learning Brief
MA3



Goals

This week:

- explain the meaning of the terms: planar graph, and face
- apply Euler's formula, $v + f - e = 2$, to solve problems relating to planar graphs
- explain the meaning of the terms: walk, trail, path, closed walk, closed trail, cycle, connected graph, and bridge
- investigate and solve practical problems to determine the shortest path between two vertices in a weighted graph (by trial-and-error methods only)

Theoretical Components

Resources:
See notes and worked examples. Also, don't forget you have access to mathspace.co

Knowledge Checklist

- Planar graphs
- Converting non-planar graphs to planar graphs
- Connected network
- 'Counting' vertices, edges and faces
- Euler's formula
- Concept of a 'walk'
- Weight edges
- Open and closed paths

Practical Components

There are questions to be answered in the booklet *Week 14/15 Exercises*

Order

1. Complete the questions in the *Booklet.*
2. Complete the Investigation below.
3. Submit the questions for marking.
4. Complete any overdue Mathspace tasks.

Investigation

See the end of the learning brief 😊

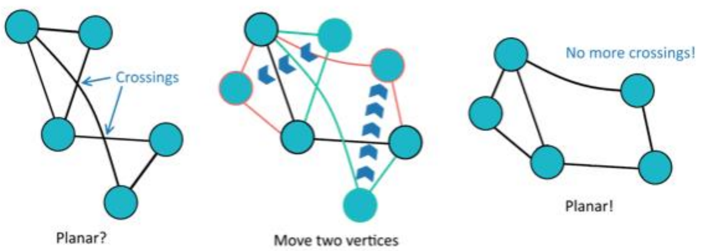
On-line Quiz

Make sure to complete any overdue Mathspace tasks.

WEEK 14 and 15 NOTES and EXERCISES

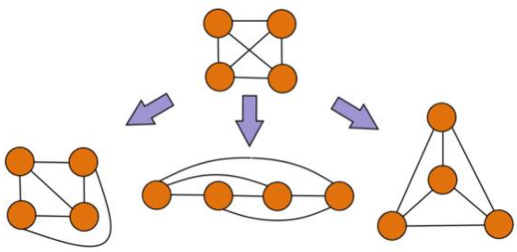
Planar Graphs

You may have noticed that networks often have their edges crossing over each other. But maybe there's a way to move the vertices into a certain configuration so that **none of the edges cross each other** anymore. Such a configuration is called a **planar representation**, and networks that have one are called **planar networks**.



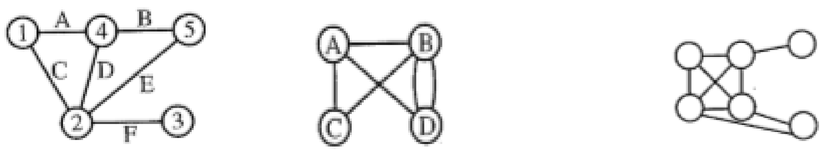
It isn't always obvious at first glance whether a network is **planar** or **non-planar** - sometimes you have to move the vertices around for a long time before none of the edges cross each other anymore.

The top network is **planar**, and below are three of its **planar representations**.



A planar graph can be defined as follows: If a graph (network) has no edges (paths) which cross, then it is a planar graph.

Consider the following graphs.

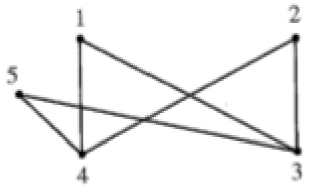


The first figure is planar as none of the paths cross each other. The next two figures are apparently not planar as there are paths that cross. Both, however, could be made planar by moving a vertex.

Converting non-planar graphs

Although it may appear that a graph is not planar, by modifying the graph it may become planar. There is no specific method but by trial and error it may be possible to remove all the crossing paths. Also, it may be possible to move vertices so that the connecting edges don't cross.

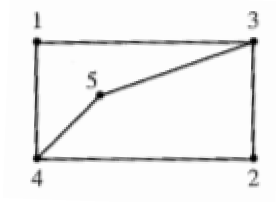
Example
Convert the graph below to a planar graph.



E = {(1, 3), (1, 4), (2, 3), (2, 4), (3, 5), (4, 5)}

Two crossings could be eliminated if vertex 2 is exchanged with vertex 3. Check that all the edges are connected to the same vertices.

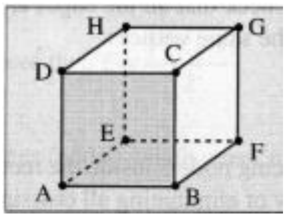
Placing node 5 inside the rectangle is one way of eliminating all crossings. Note that this planar graph is only one of several possible answers.



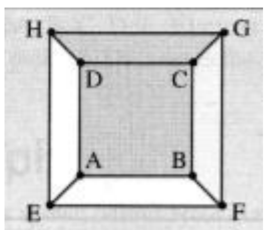
Converting three-dimensional solids to planar graphs

Example
The figure at right shows a cube with vertices
V = {A, B, C, D, E, F, H}

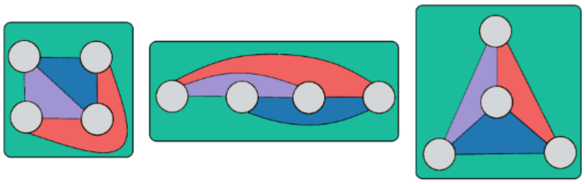
E = {(A, B), (A, D), (A, E), (B, C), (B, F),
(C, D), (C, G), (D, H), (E, F), (E, H),
(F, G), (G, H)}



Imagine the three-dimensional cube 'collapsing' to a Two-dimensional graph ie collapsing the face A-B-C-D into the face E-F-G-H
Check the edges to see if they are the same as in the original shape.

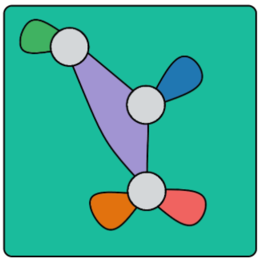


Once we have a planar representation, we can define a face (or region) as an area of the plane bounded by edges. The part of the plane outside the network is also a face.

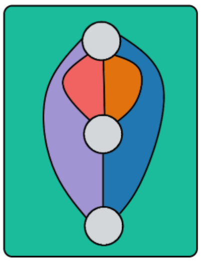


Each of these networks has four faces, three enclosed by edges and one outside.

Non-simple networks that are planar also have faces, though some of their faces are bounded by only one or only two edges.



6 faces

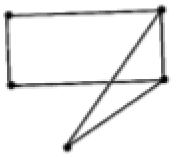


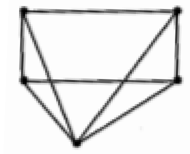
5 faces

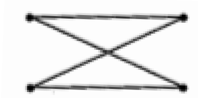
If a network is not planar, then there are no faces to define – the crossing of the edges makes it impossible.

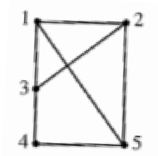
Exercise 1

1. Modify the following shapes so that their representations are planar.

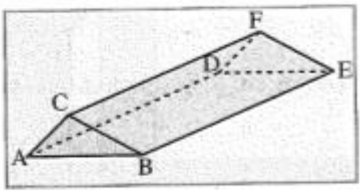
a) 

b) 

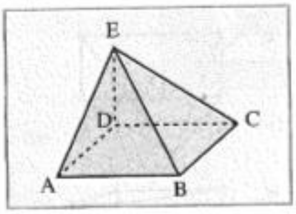
c) 

d) 

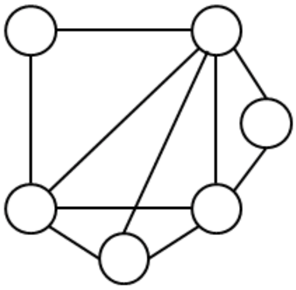
2. Convert the three-dimensional triangular prism below into a planar graph.



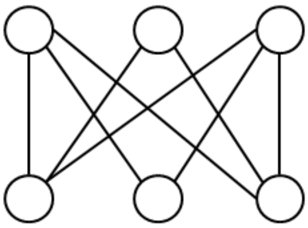
3. Convert the rectangular pyramid shown into a planar graph.



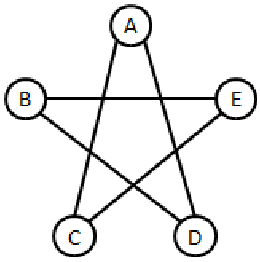
4. Convert the non-planar graph into a planar graph and determine the number of faces



5. Convert the non-planar graph into a planar graph and determine the number of faces



6. Convert the non-planar graph into a planar graph and determine the number of vertices, edges, and faces.



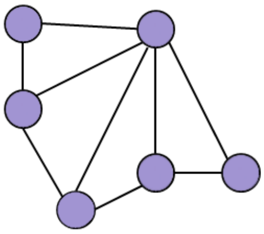
Euler's Formula

The Swiss mathematician Leonard Euler (pronounced OIL-er) was a pioneer in the mathematics of networks in the 18th century. He noticed something interesting about networks that are connected and planar, which is that the number of vertices *V*, the number of edges *E* and the number of faces *F*, satisfy the formula

$$V + F - E = 2$$

We call this Euler's formula.

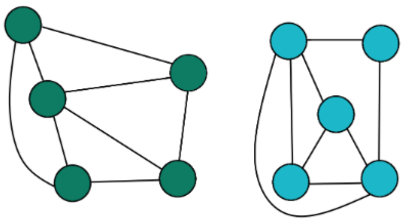
This is a planar network.



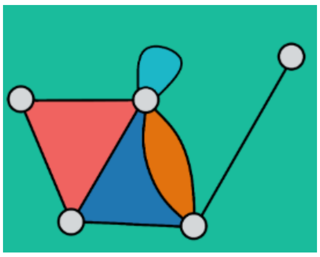
It has 6 vertices, 9 edges and 5 faces and, 6 + 5 – 9 = 2

A connected network has 5 vertices and 5 faces. How many edges does it have?
$V = 5$, $F = 5$ thus $V + F - E = 2$ which becomes $5 + 5 - E = 2$.
Rearranging gives $E = 8$

Here are two different networks that are connected, planar, and have 5 vertices, 8 edges and 5 faces.



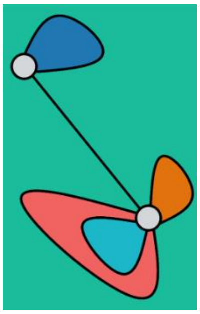
They also happen to be simple. There are many other examples, including some that aren't simple.



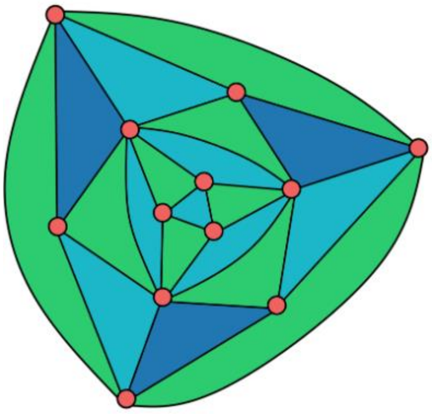
Is there a connected, planar network that has 6 vertices, 8 edges and 3 faces?
No, because 6 + 3 – 8 ≠ 2 using $V + F - E = 2$

Is there a connected, planar network that has 2 vertices, 3 edges and 3 faces?
Yes, because 2 + 3 – 3 = 2

Below is an example of a non-simple network with 2 vertices, 5 edges and 5 faces.

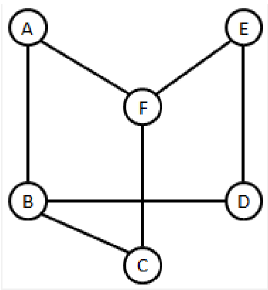


Is there a connected planar network that has 12 vertices, 30 edges and 20 faces?
Yes, as 12 + 20 – 30 = 2

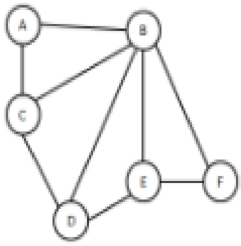


Exercise 2

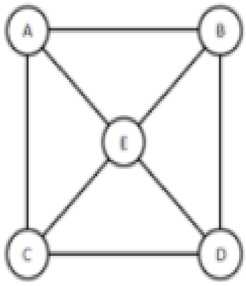
1. Convert this graph to its planar representation. Use this to verify Euler's formula.



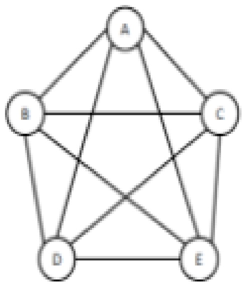
2. Consider the following graphs, converting Graphs C and D to their planar form, if possible.



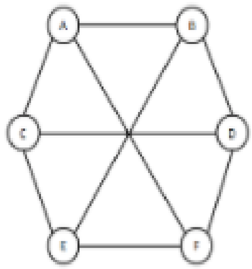
A



B



C

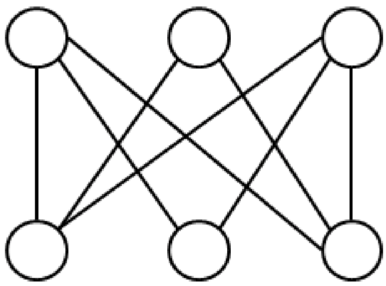


D

Now, complete the table for the above graphs:

Graph	Vertices	Faces	Edges	V + F- E
A				
B				
C				
D				

3. Convert this graph to its planar representation.



   a. What is the total number of edges, E, on this planar graph?

   b. How many vertices, V, are there?

   c. How many faces, F, are there?

   d. Use this to verify Euler's formula.

4. A connected planar graph has 14 edges and 7 vertices. Solve for F, the number of faces. (Use Euler's formula)

5. A connected planar graph has 8 vertices and 2 faces. Solve for E, the number of edges. (Use Euler's formula)

6. A connected planar graph has 3 vertices and 3 faces.
   a. How many edges does it have?

   b. Draw a planar graph that represents the description above.

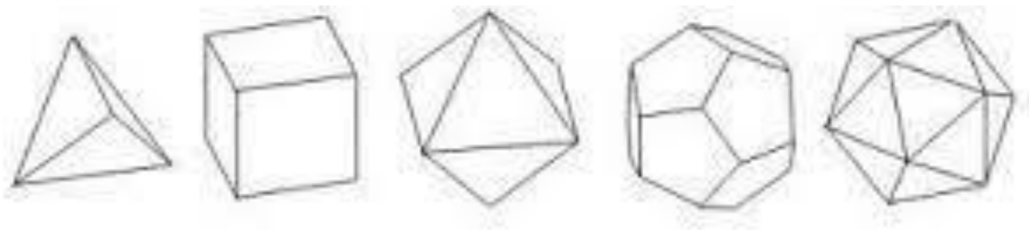
7. A graph has 7 vertices, 8 faces and 19 edges
   a. Evaluate Euler's formula for this graph.
   b. Does Euler's formula hold for this graph?
   c. Is this a planar graph?

8. Planar graphs and platonic solids

https://www.mathsisfun.com/platonic_solids.html

What is a platonic solid?

Each of these solids can be represented by a planar graph.

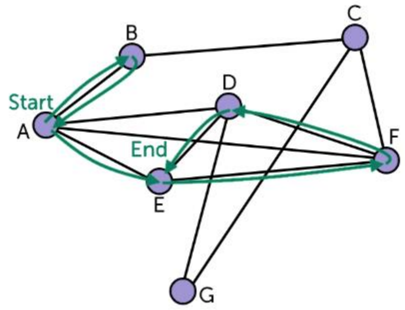


Complete the following table:

Polyhedron	Shape of each face	Faces (F)	Vertices (V)	Edges(E)	F+V- E
Tetrahedron				6	
Cube	Square				
Octahedron		8			
Dodecahedron			20		
Icosahedron				30	

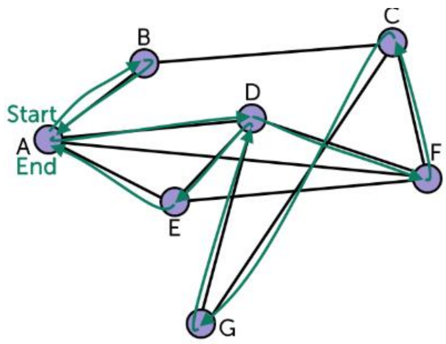
Moving around a network (Walks)

If we start at a vertex and then move along an edge (in the direction of the arrow, if it's a directed edge) to a new vertex, we are starting a **walk**. We can continue the walk by selecting another edge coming out of the second vertex, then a third edge coming out of the third vertex, and so on. In a simple network we can represent a walk by a sequence of vertices. Here is a simple, undirected network, with arrows representing a walk around the network:



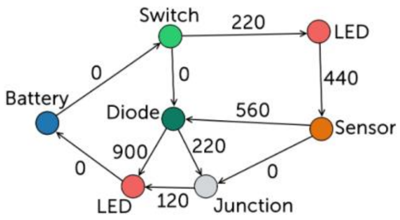
We can record this walk as the sequence ABAEFDE.

A walk that **starts and ends at the same vertex** is called **closed**, and if it **starts and ends at different vertices** it is called **open**. The walk above is open - below is a different walk on the same network:

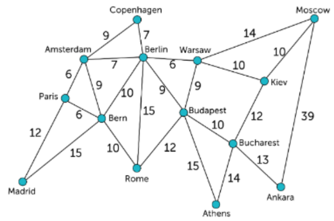


This walk has sequence *ABADFCGDEA*. Since it starts and ends at the same vertex, it is **closed.** We can also tell from just looking at the sequence (and not the diagram) that's the walk is closed, since the sequence **starts and ends with the same letter**.

We represent relationships in a network with edges, and often we will want to **attach a measurement** to each edge to show how the relationships are different. For example, if we were making a network based on the roads between towns, we might want to show more than just which towns are connected - **how long** is each road, or **how long does it take to travel** along each one? To show this kind of extra detail, we **add a number** to each edge, and call this number a **weight**. Here are a few examples.

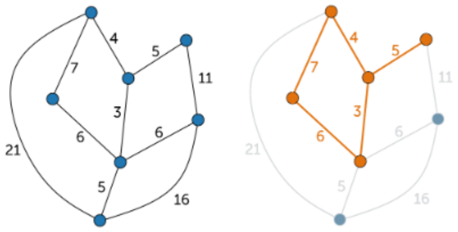


A simple circuit with resistance measured in OHMS.



Approximate travel time, in hours, between some European capital cities.

If a network has weights on its edges, we can give a weight to **the whole network** by adding up all the weights. This applies to subnetworks too. Similarly, we can give a weight to any walk by adding up all the weights on the edges chosen in the walk (the weight of a walk is sometimes called its **length**).



This **network** has a weight of 4+5+7+3+6+6+11+5+16+21=84.

The **subnetwork** in the middle (highlighted) has weight 4+5+7+3+6=25.

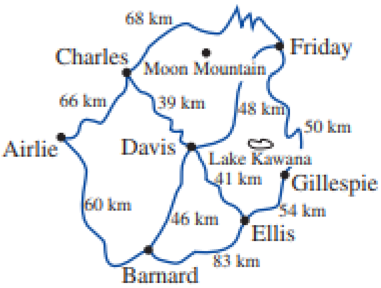
Example

George and Effie want to drive from Airlie to Gillespie using the map at below.

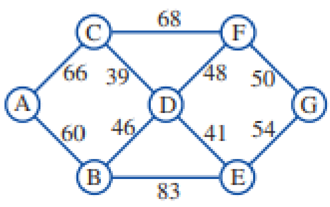
a) Draw a network which represents the map.

b) Given that each road taken must bring them closer to Gillespie, list the number of ways from Airlie to Gillespie. How many ways are possible?

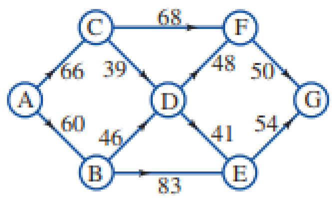
c) Identify the shortest path from the possible routes in b.



Represent towns with circles, called nodes, labelled with the first letter of the town. Ignore the bends in the roads and use straight lines to represent roads connecting the towns.



Each road taken from Airlie must go towards Gillespie. Indicate the direction on each arc with an arrow.



Use the network to list the number of ways from A to G.

A–B–D–E–G
A–B–D–F–G
A–B–E–G
A–C–D–E–G
A–C–D–F–G
A–C–F–G
There are 6 ways to go from Airlie to Gillespie.

Add the lengths of the nodes to calculate the distances of the 6 routes

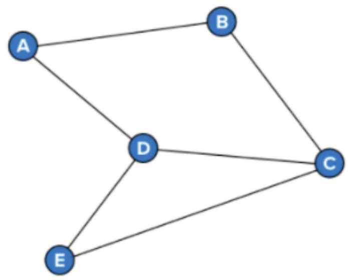
A–B–D–E–G (60 + 46 + 41 + 54)	201 km
A–B–D–F–G	204 km
A–B–E–G	197 km
A–C–D–E–G	200 km
A–C–D–F–G	203 km
A–C–F–G	184 km

The shortest path is A–C–F–G: Airlie to Charles to Friday to Gillespie.

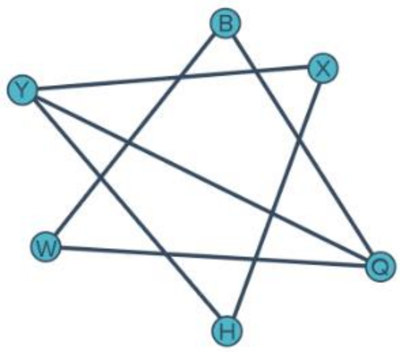
Effie consults the local travel adviser about the travel times for the stages in the journey planned in the example above. She could then redraw the network with the average time (in minutes) taken to drive between the towns to determine which path would take the least time and what is that time?

Exercise 3

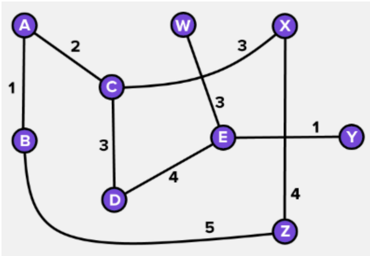
1. Determine a valid walk for this network



2. Determine a closed walk for the graph, starting from H with 3 edges.



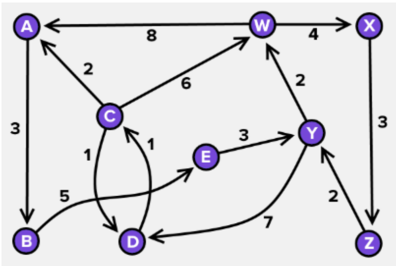
3. Find the weights of the following paths.



a) Path A-B-Z-X

b) Path X-C-A-B-Z

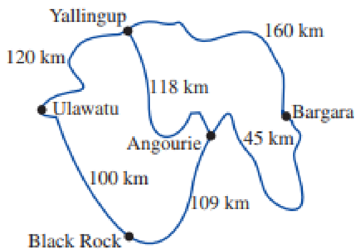
4. Find the weights of the following paths in this network.



a) The path from *Z* to *C*

b) The path from *B* to *X*

5. A traveller plans a journey from Ulawatu to Bargara (shown on the road map below).



a) Draw a network to represent this situation.

b) Calculate the longest path if no road is travelled twice.

c) The travelling times between each town are:
Ulawatu–Yallingup 85 min
Ulawatu–Black Rock 75 min
Yallingup–Angourie 80 min
Black Rock–Angourie 82 min
Yallingup–Bargara 120 min
Angourie–Bargara 34 min.

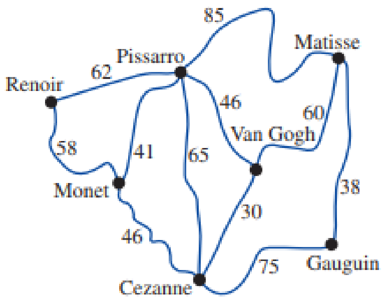
Draw a network of this situation showing the time taken to travel between towns on each edge of the network.

d) Calculate the longest time taken to travel from Ulawatu to Bargara.

e) Complete the table showing the shortest distance between each of the towns.

	Ulawatu	Yallingup	Black Rock	Angourie	Bargara
Ulawatu	——	120	100	209	
Yallingup	——	——	220		
Black Rock	——	——	——		
Angourie	——	——	——	——	

6. A traveller plans a journey from Renoir to Gauguin. The distances between various nearby towns are shown on the map at right.



a) Calculate the shortest path.

b) The travelling times between the following towns are:

Renoir–Pissarro	47 min
Renoir–Monet	44 min
Monet–Cezanne	40 min
Pissarro–Cezanne	45 min
Pissarro–Van Gogh	34 min
Pissarro–Matisse	75 min
Pissarro–Monet	25 min
Cezanne–Van Gogh	20 min
Van Gogh–Matisse	38 min
Cezanne–Gauguin	59 min
Matisse–Gauguin	28 min

i) Draw a network of this situation showing the time taken to travel between towns on each arc of the network.

ii) Calculate the longest time (without taking any paths twice) to travel from Renoir to Gauguin.

iii) Calculate the shortest time to travel from Renoir to Gauguin.

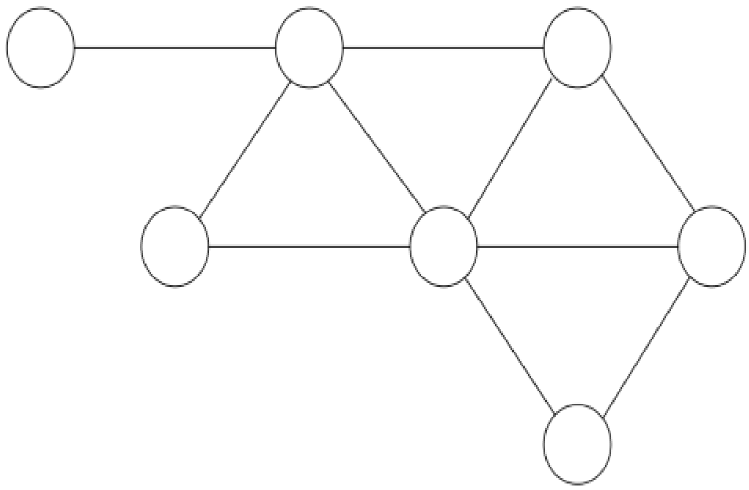
iv) Complete the table below showing the shortest distance between each of the towns.

	Renoir	Pissarro	Monet	Cezanne	Van Gogh	Matisse	Gauguin
Renoir	——						179
Pissarro	——	——	41				123
Monet	——	——	——				
Cezanne	——	——	——	——			
Van Gogh	——	——	——	——	——		
Matisse	——	——	——	——	——	——	

Task 1

Here is a graph showing a group of people. You need to work out who's who using the clues provide below.
Bella and Ciara are friends. Erin and Ciara are not friends. Bella is Fiona's only friend. Anna has more friends than anyone else. Daphne has three friends. Gill and Daphne are not friends and Erin has two friends.

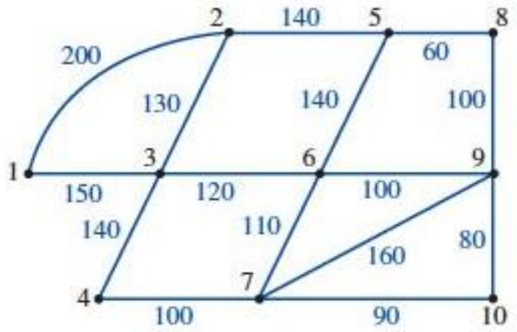
a)



b) Construct an adjacency matrix to represent this graph.

Task 2

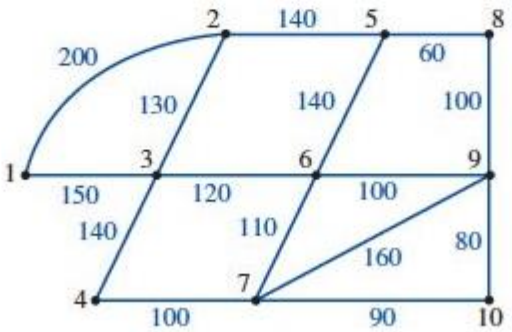
Recyclable materials are to be collected from households in a part of a suburb. The network below represents this area, where the vertices are street intersections, and the edges are the streets. The numbers indicated the length (in metres) of the streets between intersections.

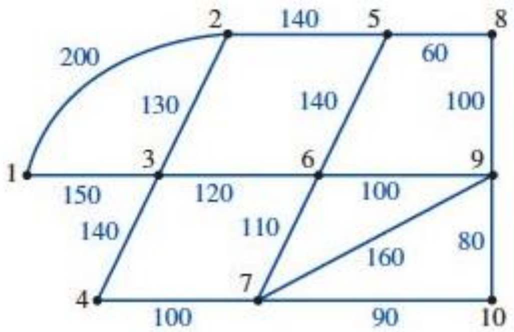


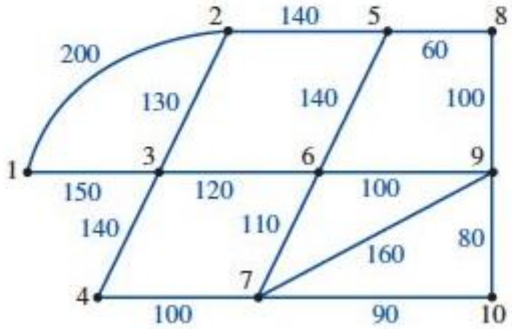
1. Does this network represent a planar graph? Explain your answer.

2. Verify Euler's formula for this network.

3. A taxi driver needs to transport a customer from 1 to 10. Find the shortest distance for them to travel between intersections 1 and 10. (Clearly label/demonstrate the routes taken to ensure you have found the shortest distance)



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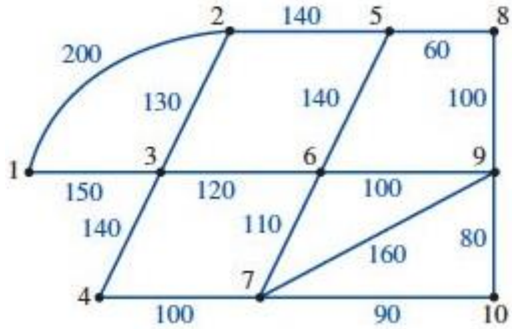
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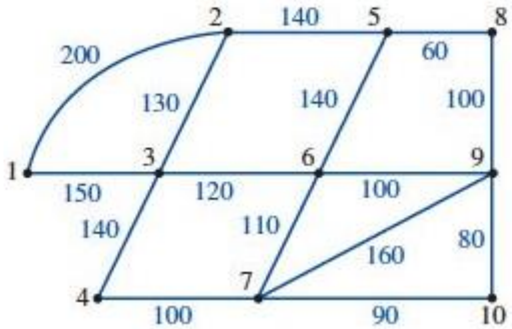
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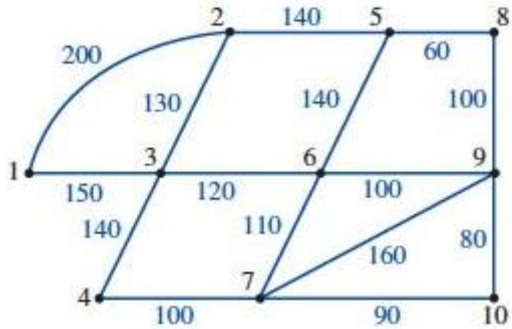
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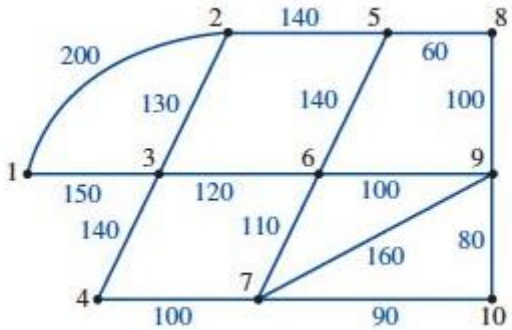
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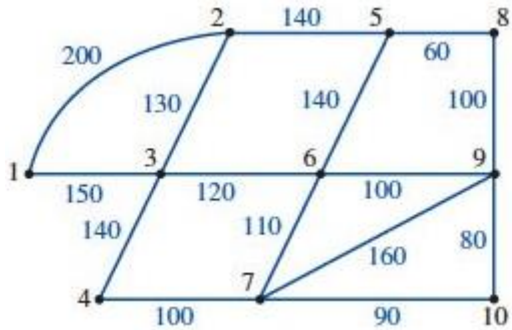
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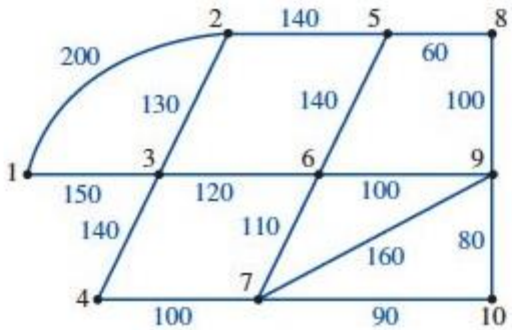
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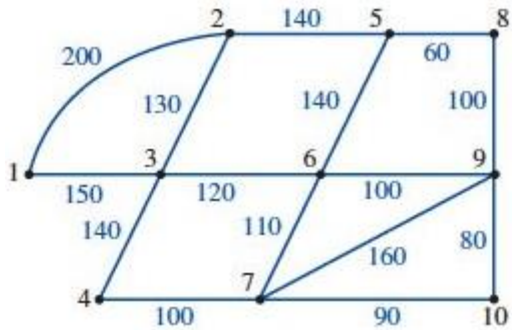
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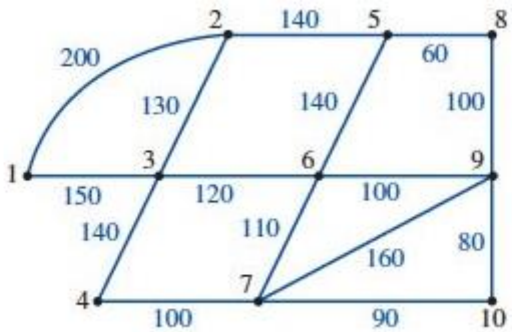
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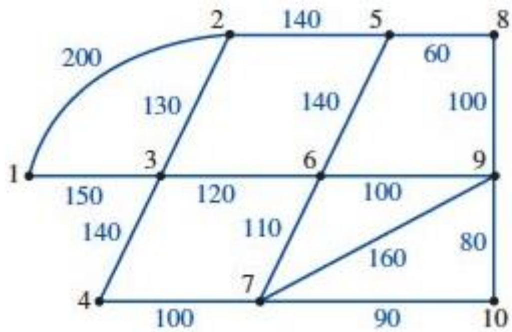
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Marking Rubric **Week 14/15** **Name:**

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work, including exercises and Mathspace task, of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation Task	Student completes the investigation task of the week to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and Communications	Student responses are accurate and appropriate in presentation of mathematical ideas, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and Techniques	Student submitted work selects and applies appropriate mathematical techniques to solve practical problems and demonstrates proficiency in the use of mathematical facts, techniques, and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises, Mathspace/online task, and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
				FINAL	**/20**

Student Reflection: How did you go with this week's work? What did you learn? What did you find easy?

What do you need to work on?