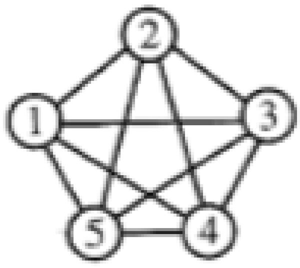
Week 13
Term 2
2022

HAWKER COLLEGE
Engage | Inspire | Achieve

Learning Brief

MA3

Goals



This week:

- explain the meanings of the terms: graph, edge, vertex, loop, degree of a vertex, subgraph, simple graph, complete graph, bipartite graph, directed graph (digraph), arc, weighted graph, and network
- identify practical situations that can be represented by a network, and construct such networks; for example, trails connecting camp sites in a National Park, a social network, a transport network with one-way streets, a food web, the results of a round-robin sporting competition
- construct an adjacency matrix from a given graph or digraph.

Theoretical Components

Resources:

Introduction to graphs and networks:
https://mathspace.co/textbooks/syllabuses/Syllabus-1023/topics/Topic-20211/subtopics/Subtopic-279440/?activeTab=theory

Network representations:
https://mathspace.co/textbooks/syllabuses/Syllabus-1023/topics/Topic-20211/subtopics/Subtopic-266109/?activeTab=theory

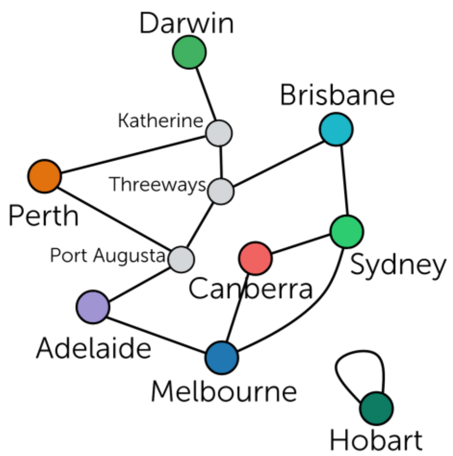
This clip covers several of the concepts in this week's booklet.
https://www.youtube.com/watch?v=i7b47wFtWxw

Knowledge Checklist

- Definition of a network
- Definition of vertex (node) and edge (path)
- Degree of a vertex
- Labelling vertices and edges
- Constructing networks
- Matrix representation of networks

Practical Components

There are questions to be answered in the booklet *Week 13 Exercises*



Investigation

On HawkerMaths and attached to this week's work

On-line Quiz

None for this brief

WEEK 13 NOTES and EXERCISES

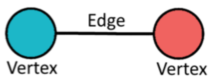
Introduction to graphs and networks

Networks are maps that can represent an amazing variety of different things: simplified maps, relationships between people, sub-tasks in a building project, computer terminals or the flow of traffic through a city. In each case the network provides a means of studying real-life situations so that decisions can be made. When drawing a network, irrelevant information, such as bends in the roads of a map, is ignored.

A network is a collection of objects connected to each other in some specific way.

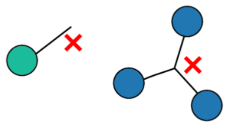
In the case of a road map the objects are the towns, while the connections are roads. In relationships the objects are parents, grandparents, cousins, aunts, etc.

The mathematical term for these objects is a *node* or *vertex*.



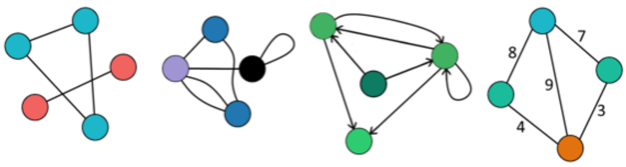
This is probably the simplest of networks and consists of two vertices and one connection between them. This connection is called an *edge* or *path*.

An edge must be drawn between *exactly two* vertices. Neither of these are valid edges:



An edge can't connect at only one end and can't connect more than two vertices together.

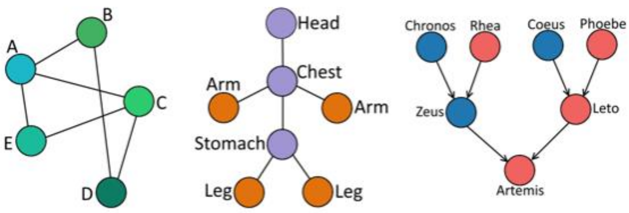
So, a **network** (sometimes called a **network graph**) is a collection of vertices with edges drawn between them.



You can notice a few things about these networks:

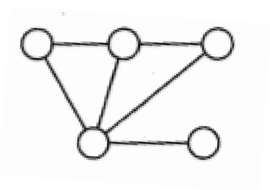
- Some edges have *direction* (they look like arrows), and some *don't*.
- Sometimes a vertex is *connected to itself* by an edge.
- Sometimes vertices are connected to another vertex by *more than one edge*.
- One of the networks has *numbered edges*.

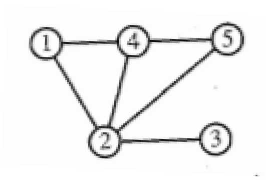
Here are three more networks with extra detail - these have *letters* or *words* for each vertex.



The first distinguishing feature of a network are the total number of vertices and total number of edges.

Example: Below is a simple network.





By labelling the vertices with numbers we can see that there are 5 vertices and counting the connections gives 6 edges.

The degree of a vertex

Each network may have a number of edges connecting it with the rest of the network. This number is called the degree. To determine the degree of a vertex simply count its edges. The table below shows the degree of each vertex in the network above.

Vertex	**1**	**2**	**3**	**4**	**5**
Degree	**2**	**4**	**1**	**3**	**2**

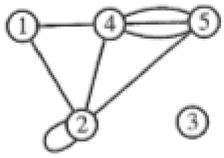
A vertex with degree 0 is not connected to any other vertex and is called an *isolated vertex*.

An edge which connects a vertex to itself is called a *loop* and contributes 2 towards the degree.

If two (or more) edges connect the same pair of vertices they are called *parallel edges* and all count towards the degree. Otherwise, if there is only one connection between two vertices, the connection is called a *simple*, or *single*, connection.

Example

Determine the degree of each vertex (node) in the figure below.



Node 1: Has 2 simple edges.	Degree of node 1 = 2
Node 2: Has 3 simple edges and 1 loop.	Degree of node 2 = 3 + 2 = 5
Node 3: Has no edges — an isolated node.	Degree of node 3 = 0
Node 4: Has 2 simple edges and 3 parallel edges.	Degree of node 4 = 2 + 3 = 5
Node 5: Has 1 simple edge and 3 parallel edges.	Degree of node 5 = 1 + 3 = 4

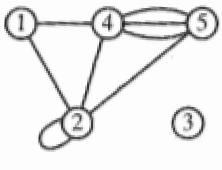
Representation of networks

To label vertices, simply list them. If there are three vertices labelled A, B and C write V ={A, B,C}

If an edge connects vertex 1 with vertex 3, we represent the edge as (1, 3). If there is a loop at vertex 4, its edge is (4,4). If there are 2 parallel edges between vertices 2 and 4, we write (2, 4), (2, 4).

Example

Label the vertices and edges for the figure shown.



Label the vertices V ={1, 2, 3, 4, 5}

Examine each edge, in turn.

Vertex 1–vertex 4	(1, 4)
Vertex 1–vertex 2	(1, 2)
Vertex 2–vertex 2 (loop)	(2, 2)
Vertex 2–vertex 4	(2, 4)
Vertex 2–vertex 5	(2, 5)
Vertex 4–vertex 5 (3 parallel edges)	(4, 5), (4, 5), (4, 5)

Combine vertices and edges into a list:

$V = \{1, 2, 3, 4, 5\}$
$E = \{(1, 4), (1, 2), (2, 2), (2, 4), (2, 5), (4, 5), (4, 5), (4, 5)$

Some points about this representation.

- There is no '3' in the list of edges *E*. This implies it is an isolated vertex.
- The number of pairs in *E* = 8 which must be the same as the number of edges.
- The number of times a vertex appears anywhere inside *E* equals the degree of the vertex eg the digit 4 appears 5 times, so the degree of vertex 4 = 5.

Example

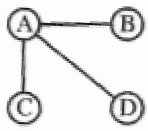
Construct a graph (network) from the following list of vertices and edges.

$V = \{A, B, C, D, E\}$
$E = \{(A, B), (A, C), (A, D), (B, C), (B, D), (B, D), (C, E), (D, E), (E, E)\}$

1 Start with a single vertex, say vertex A, and list the vertices to which it is connected.

Vertex A connected to B, C and D.

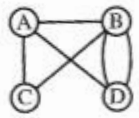
2 Construct a graph showing these connections.



Take the next vertex, say B, and list the vertices to which it is connected.

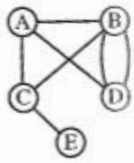
Vertex B is connected to A (already done), C and D (twice: parallel edge).

4 Add the edges from step 3.



5 Repeat steps 3 and 4 for vertex C.

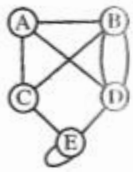
Vertex C is connected to A (already done), B (already done) and E.



6 Repeat steps 3 and 4 for vertex D and, finally, add the loop (E, E).

As a check, count the edges in the list *E* (9) and compare it with the number of edges in your final graph.

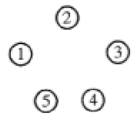
Vertex D is connected to A (already done), B (already done) and E.
Vertex E is connected to C (already done), D (already done) and E (loop).



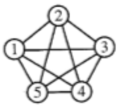
Definitions

Complete and degenerate graphs

A graph with no edges is called a degenerate or a null graph. A graph where all vertices are connected directly to all other vertices is called a *complete* graph.



A degenerate graph

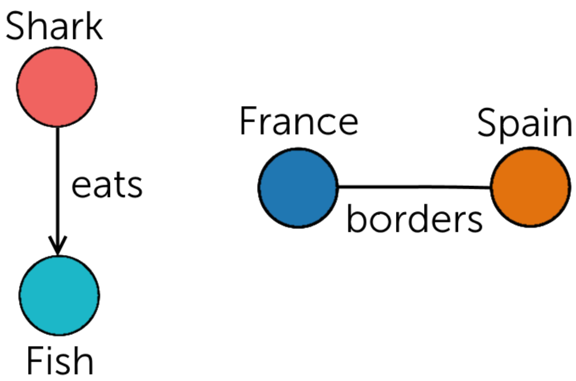


A complete graph

Directed graphs

Edges can represent a one-way connection or a two-way connection.

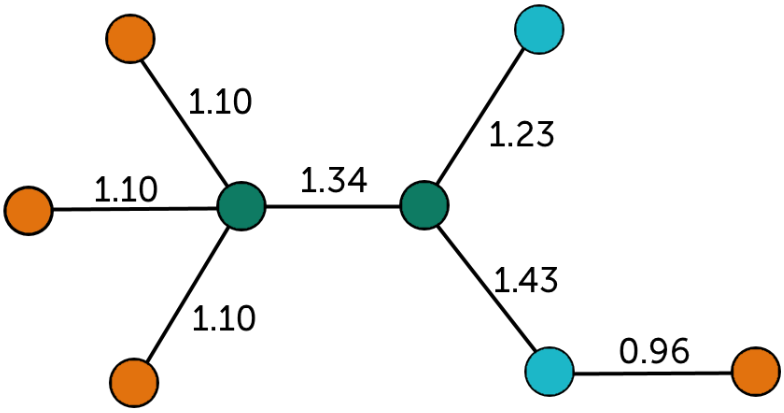
- Directed edges (also called arcs) are drawn as arrows and represent a one-way connection.
- Undirected edges are represented by lines and represent a two-way connection.



*The first graph has a **directed edge** because sharks eat fish and not the other way around - a one-way relationship.*
The second graph has an undirected edge because France and Spain share a border - a two-way relationship.

Weighted graphs

We have seen that the edges of a graph can be given numerical values like the graph below:

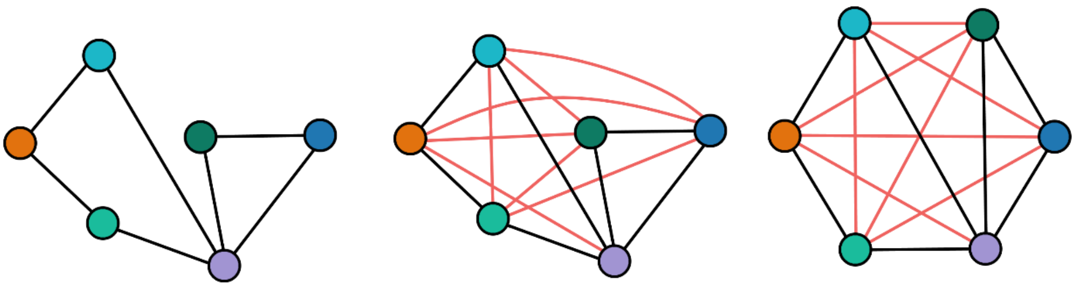


These values are called weights and can represent some property such as distance or cost. When the edges of a graph have weights we refer to the graph as a weighted graph. The **weight of the graph** is the sum of the weights of all edges.

Subgraph

A graph is called a subgraph (or subnetwork) when all vertices and edges are also the vertices and edges of another graph.
If we take a graph and delete some edges, or some of its vertices (and all edges connected to it), we obtain a subgraph of the original.

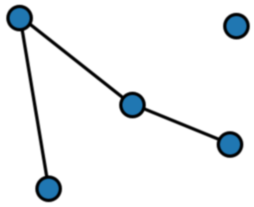
For example, any simple graph that has *n* vertices is a subgraph of the complete graph with *n* vertices - we can just add or delete edges to get from one to the other:



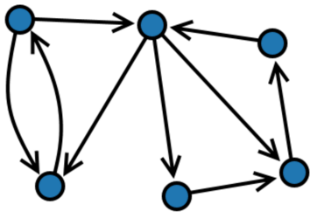
The graph on the left has the red edges added in, and then the vertices moved a little.
We end up with a complete graph.

Exercise 1

1. How many vertices and edges does this network have?



2. How many vertices and edges does this network have?



3. Construct a network from the following list of vertices and edges:

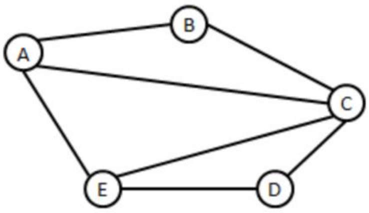
   a. V = {1, 2, 3, 4} E = {(1,2), (1,4), (2,3), (3,4)}
   b. V = {A, B, C, D, E} E = {(A,B), (A,C), (A,C), (B,B), (B,C), (B,D), (C,D)}
   c. V = {1, 2, 3, 4, 5, 6} E = {(1,2), (1,4), (1,6), (2,3), (2,6), (3,4), (4,6)}

4. Six people in a room shake hands with each other.

a) Represent the handshakes as a complete graph.

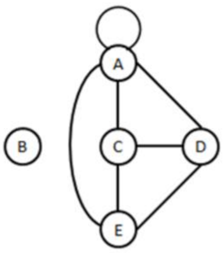
b) How many handshakes are there?

5. Consider the following graph:



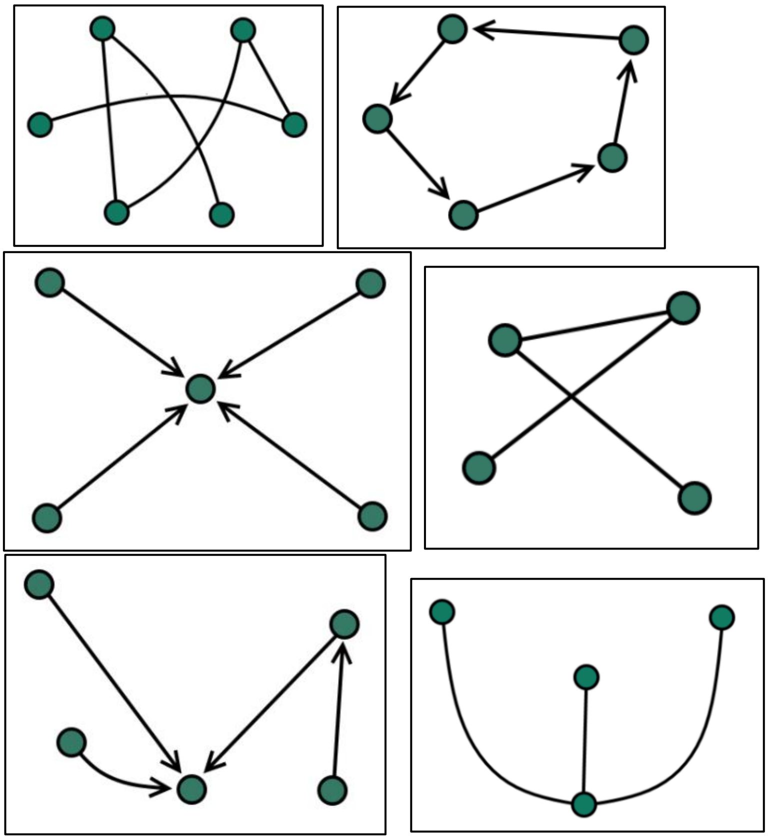
a. How many edges are connected to the vertex A?
b. Which vertices are adjacent to the vertex E?
c. Which vertex is directly connected to all other vertices?
d. Which vertex in not directly connected to vertex A?

6. Consider the following graph:

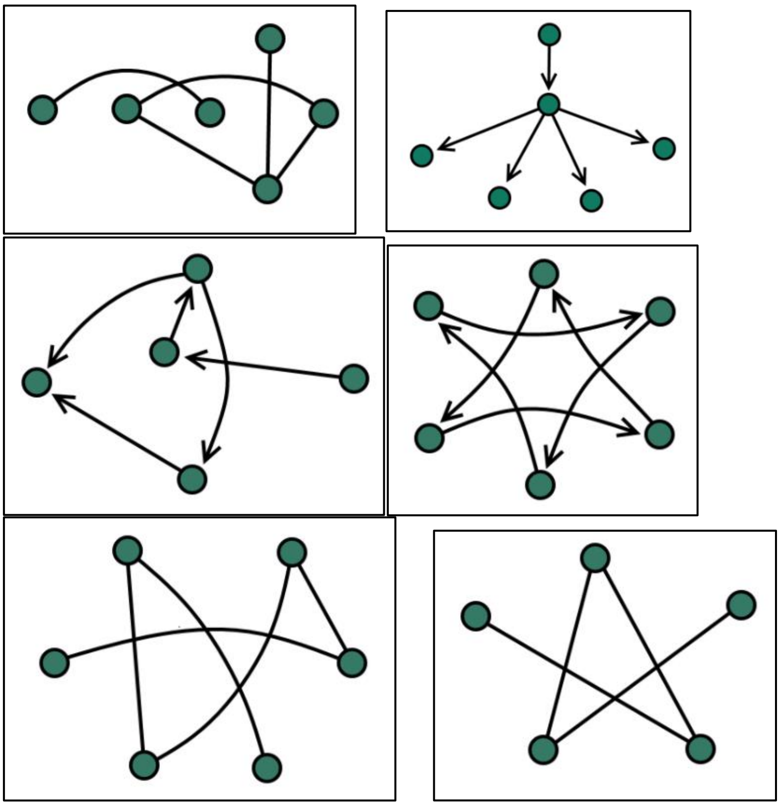


a. Which vertex is isolated?
b. How many edges are there in the network?
c. How many vertices are there?
d. Which vertex is the loop connected to?

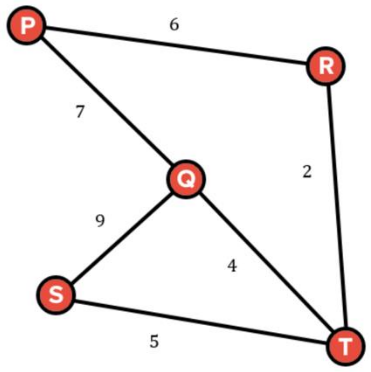
7. Tick all directed networks:



8. Tick all undirected networks:



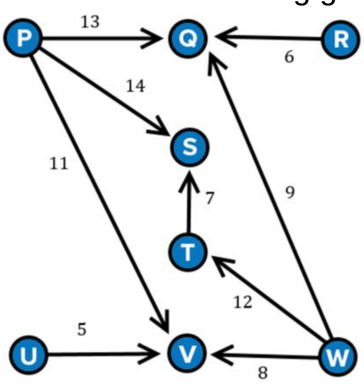
9. *Consider the following network:*



a. What is the weight of the edge connecting Q and S?

b. What is the weight of the entire network?

10. Consider the following graph:



a. What is the weight of the subgraph that has only the vertices P, S, and V, and any edges between them?

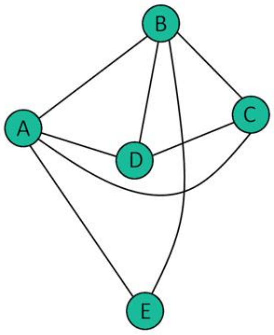
b. What is the weight of the subgraph formed by deleting vertices V, W, T, and S, along with all edges connected to these vertices?

Network Representations

It is sometimes useful to use a table (its proper name is the adjacency matrix) to represent a network. This is a square array of numbers, one row and column for each vertex, that records how many connections there are between the vertices. For an undirected network, the entry for a particular row and column represents the number of edges connecting the matching vertices.

Examples:

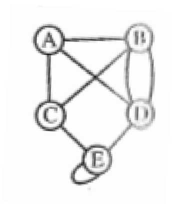
1. Here's a simple network with 5 vertices and its matrix representation:



$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array} \begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

- The diagonal running top-left to bottom-right (called the main diagonal) has only 0s.
  This represents how no edge is connected to itself in the network - there are no loops.
- The numbers not on the main diagonal are only 0s and 1s.
  This represents how no vertex is connected to any other by more than one edge.

2. Represent the network shown as a matrix:



The completed matrix is:

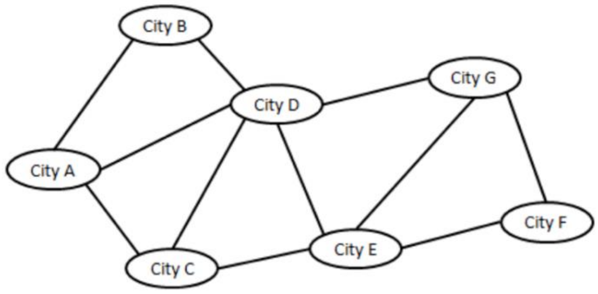
	A	B	C	D	E
A	0	1	1	1	0
B	1	0	1	2	0
C	1	1	0	0	1
D	1	2	0	0	1
E	0	0	1	1	2

Some points to note:

- While A to A is 0, E to E is 2 as there is a *loop*
- The sum of a row (or column) gives the degree of that vertex
- If an entire row and its corresponding entire column has only 0s then that vertex is isolated

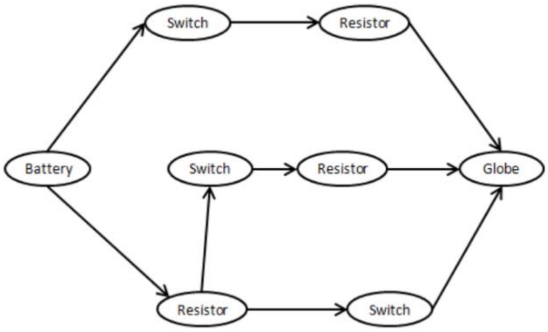
Exercise 2

1. This diagram represents the internet fibre optics connecting several cities.



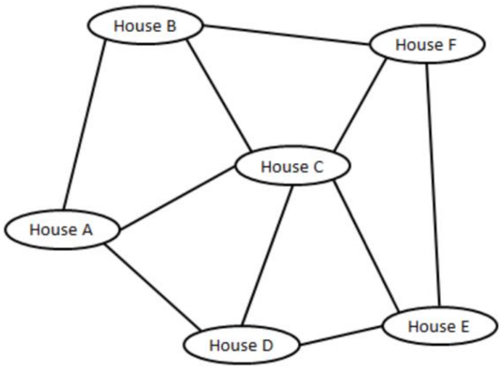
   a. What do the edges represent?
   b. How many vertices are there?
   c. How many edges are there?

2. This diagram represents an electrical circuit that includes a globe resistors and multiple switches.



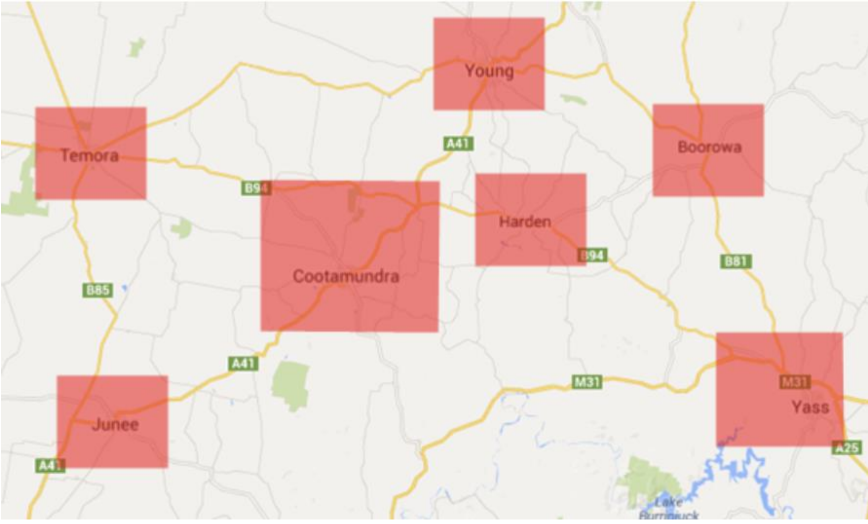
   a. What do the edges represent?
   b. How many vertices are there?
   c. How many edges are there?

3. The following figure displays newspaper routes between several houses.

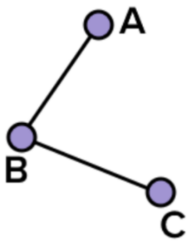


   a. What do the edges represent?
   b. How many vertices are there?
   c. How many edges are there?

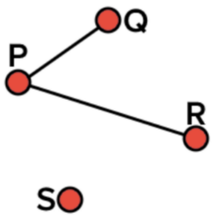
4. The given map shows major roads connecting various towns. The shaded regions show how far each town extends. Draw a network to represent the major roads connecting towns on the map. (major roads are labelled, e.g. A41)



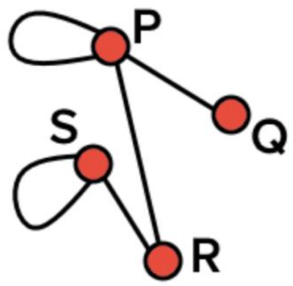
5. Create a matrix for the following network.



6. Create a matrix for the following network.



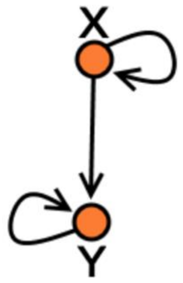
7. Create a matrix for the following network.



8. Draw a network for the following matrix.

$$\begin{array}{c} \\ P \\ Q \\ R \\ S \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} P & Q & R & S & T \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

9. Create a matrix for the following network.



10. Draw a network for the following matrix.

$$\begin{array}{c} \\ X \\ Y \\ Z \end{array} \begin{array}{c} \begin{array}{ccc} X & Y & Z \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array}$$

Investigation Week 13

Consider a network of 4 vertices, where each vertex is connected to each of the other 3 vertices with a single edge. This is a complete graph.

Question 1:

a. Construct a diagram of the network described above.
b. How many vertices and edges are there?
c. Construct a matrix representation for the network you have drawn.

Question 2: Repeat Question 1 for a network of:

a. 5 vertices
b. 8 vertices

Question 3: if you compare the results from Question 1 and 2, you will notice a simple relationship between the number of vertices and the number of edges.

Use this relationship to predict the number of edges for similar networks of:

a. 10 vertices
b. 20 vertices
c. 100 vertices

Make sure you explain how you calculated your answers. You can use the following blank page to complete your work.

Note: The increase in the number of edges is one of the problems that had to be overcome in the design of computer networks.

Marking Rubric **Week 13** **Name:**

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work, including exercises and Mathspace task, of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation Task	Student completes the investigation task of the week to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and Communications	Student responses are accurate and appropriate in presentation of mathematical ideas, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and Techniques	Student submitted work selects and applies appropriate mathematical techniques to solve practical problems and demonstrates proficiency in the use of mathematical facts, techniques, and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises, Mathspace/online task, and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
				FINAL	**/20**

Student Reflection: How did you go with this week's work? What did you learn? What did you find easy?
What do you need to work on?