Week 9/11
Term 1
2021

HAWKER COLLEGE

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Learning Brief

MA3

Goals

$$t_{n+1} = rt_n + d$$

This brief we are covering:

- compound interest (appreciation)
- depreciation straight line depreciation
- reducing balance depreciation
- use arithmetic, geometric, and first-order linear recurrence to model and analyse practical problems

Theoretical Components

Resources:
For this week the theory work is in the *PDF file:* Week *9* and 11 Notes & Exercises

Knowledge Checklist

- Concept of compounding
- Term of investment or loan
- Interest rate per period
- Understand which applications is arithmetic progression
- Understand which applications is geometric progression
- Straight line depreciation
- Reducing balance depreciation
- $1 - \frac{r}{100}$ where r is less than 1
- Decay as a decrease in value
- Difference equations as a recurrence relation
- Using a spreadsheet
- Effect of $-1 \leq r \leq 1$ in a difference equation

Practical Components

There are questions to be answered in the booklet *Week 9 and 11 Notes & Exercises*

All overdue mathspace tasks, booklets, and investigations should be completed and submitted by Thursday of Week 9.

Investigation

See the end of the brief😊

Quiz

Compound Interest

Consider the case where a bank pays compound interest of 5% per annum on an amount of $20000. The amount is invested for 4 years and interest is calculated yearly. Compound interest receives its name because the interest which is earned is paid back into the account so that the next time interest is calculated, it is calculated on an increased amount. There is a compounding effect on the money in the account. If we calculated the amount in the account mentioned above each year, we would have the following amounts.

Start $20000
After 1 year $20000 × 1.05 = $21000
After 2 years $20000 × 1.05 × 1.05 = $22050
After 3 years $20000 × 1.05 × 1.05 × 1.05 = $23152.50
After 4 years $20000 × 1.05 × 1.05 × 1.05 × 1.05 = $24310.13

The amounts 20000, 21000, 22050, 23152.50, 24310.13, ... form a geometric sequence where a = 20000 and
r = 1.05.

We need to be a little careful, however, in using the formula $t_n = ar^{n-1}$ in calculating compound interest. This is because the original amount in the account, that is, $20000, in terms of the geometric sequence would be referred to as t_1 or *a*. In banking terms, t_1 would represent the amount in the account after the first lot of interest has been calculated and added in. To be clear and to be safe, it is best to use the following formula for compound interest.

$A = PR^n$
where $R = 1 + \frac{r}{100}$
A = amount in the account, $
P = principal, $
r = interest rate per period (that is, per year or quarter etc.), %
n = the number of periods during the investment.

Example 1

Helen inherits $60000 and invests it for 3 years in an account which pays compound interest of 8% per annum compounding each 6 months.

a) What will be the amount in Helen's account at the end of 3 years?

b) How much will Helen receive in interest over the 3-year period?

1. This is an example of compound interest.
   Use $A = PR^n$, where $R = 1 + \frac{r}{100}$. Interest is calculated each 6 months so, over 3 years, there are 6 periods: $n = 6$. Interest is 8% per year or 4% per 6 months.
   So, $r = 4\%$.
2. Write your answer.

a $P = 60\,000$
$n = 6$ half years
$r = 4\%$ per half year

So, $R = 1 + \frac{4}{100}$
$= 1.04$
$A = PR^n$
$= 60\,000(1.04)^6$
$= 75\,919.14$

At the end of 3 years, Helen will have a total amount of \$75 919.14.

1. Interest equals the amount in the account at the end of 3 years, less the amount in the account at the start of the investment.
2. Write your answer.

b Interest = Total amount – Principal
= \$75 919.14 – \$60 000
= \$15 919.14

Amount of interest earned over 3 years is \$15 919.14.

Example 2

Jim invests \$16000 in a bank account which earns compound interest at the rate of 12% per annum compounding every quarter. At the end of the investment, there is \$25616.52 in the account. For how many years did Jim have his money invested?

1. We know the value of A, P, r and R. We need to find n using the compound interest formula.

$A = 25\,616.52$
$P = 16\,000$
$r = \frac{12}{4}$
$= 3\%$ per quarter
and so $R = 1 + \frac{3}{100}$
$= 1.03$
Now, $A = PR^n$
So, $25\,616.52 = 16\,000(1.03)^n$
$1.601 = 1.03^n$

2. Try some different values of n.

Let $n = 5$ $1.03^5 = 1.159$
Let $n = 10$ $1.03^{10} = 1.344$
Let $n = 15$ $1.03^{15} = 1.558$
Let $n = 16$ $1.03^{16} = 1.605$

3. Write your answer.

It will take 16 periods where a period is 3 months. So, it will take 48 months or 4 years.

Exercise 1

Q1. \$13000 is invested in an account which earns compound interest of 8%, compounding quarterly.

a) After 5 years, how much is in the account?

b) How much interest was earned in that period?

Q2. $10000 is invested in an account which earns compound interest of 10% per annum. Find the amount in the account after 5 years if the interest is compounded monthly.

Q3. $10000 is invested in an account which earns compound interest of 10% per annum. Find the amount in the account after 5 years if the interest is compounded daily. Compare your answer to that in the previous question.

Q4. In an account earning compound interest of 8% per annum compounding quarterly, an amount of $6000 is invested. When the account is closed, there is $7609.45 in the account. For how many years was the account open?

Q5. Helena receives $15627.12 after closing an investment account which earned compound interest of 9% per annum compounding every 6 months. If Helena originally deposited $12000 in the account, for how long was it in the account?

Depreciation

W have looked at compound interest, a situation where amounts APPRECIATE, or increase in value. However, items may also decrease in value- this is known as DEPRECIATION. Cars (usually), machinery, computers and electronics are examples of things that depreciate. In other words, they're worth less over time than when you bought them.

Straight Line Depreciation

Straight line depreciation is similar to simple interest in that the depreciation (amount lost) is the same each year.

Reducing Balance Depreciation

The more common form of depreciation is reducing-balance depreciation. The rules for calculating this kind of depreciation are similar to calculating compound interest. The formula is just slightly different. The depreciation formula is:

$A = PR^n$ where $R = 1 - \frac{r}{100}$

Example

A microwave that costs $400 depreciates at 20% pa. Thus $R = 1 - \frac{20}{100} = 0.8$

Thus $A = 700\,(0.8)^n$

a) What is it's value after 3 years?

   $A = 700\,(0.8)^3 = \$358.40$

b) How long will it take for the microwave to be worthless? Let's assume 'worthless' means less than $1.

Thus $1 = 700\,(0.8)^n$ By trial and error n = 30 (value $0.80)

Exercise 2

Q1. James purchased a $18,900 motorbike, which depreciates at a compounded rate of 15% p.a.

a) What is the amount of the depreciation for the first year?

b) What is the expected value after the first year?

c) What is the depreciation for the second year?

d) What is the expected value after the second year?

e) What is the total depreciation over the two years?

f) What is the percentage of the original value remaining after two years?

Q2. Sally purchased an iPod for $800, which depreciates at 15% p.a. What is its resale value after 4 years?

Q3. The government wants to decrease its spending on job creation. Currently it is spending \$160 million and will decrease it by 6.15% p.a. over the next \$10 years. Calculate the government's spending in \$10 years' time. Round your answer to the nearest dollar.

Q4. A netbook depreciated by 28% p.a and was valued at \$900 after 7 years. What was the original price?

Difference Equations

A *recurrence relation*, also called a *difference equation*, is a rule that specifies a particular term in a sequence using the previous term or terms.

Consider the sequence, 3, 8, 13, 18,……

The first term is 3 and each subsequent term in the sequence is 5 more than the previous term.

We can write this as $t_{n+1} = t_n + 5$, $t_1 = 3$

This is called a first order difference equation because it links consecutive terms in the sequence.

$t_{n+2} = t_{n+1} – t_n$; $t_1 = 6$, $t_2 = 9$ is an example of a second order difference equation as the previous two terms are required to find the next term. The Fibonacci sequence is an example of a second order difference equation.

The general *first order recurrence relation* is given by

$t_{n+1} = rt_n + d$ with $t_1 = a$ (first term)

Note: For an arithmetic sequence $r = 1$ and for a geometric sequence $d = 0$

The sequences we will be dealing with here are neither arithmetic or geometric.

We will use spreadsheets to do the calculations for us.

Example

Use a spreadsheet to generate 20 values for the sequence, arising from the difference equation $t_{n+1} = 0.8t_n + 1.2$; $t_1 = 0.2$.

The completed spreadsheet looks like this.
Note that you only fill in Row 4 and then use the Fill Down feature.

This is neither an arithmetic or geometric sequence and there is no common difference and no common ratio.

	A	B
1	Worked Example 14	
2	n	t_n
3	1	0.2
4	=A3+1	=0.8*B3+1.2
5	=A4+1	=0.8*B4+1.2
6	=A5+1	=0.8*B5+1.2
7	=A6+1	=0.8*B6+1.2
8	=A7+1	=0.8*B7+1.2
9	=A8+1	=0.8*B8+1.2
10	=A9+1	=0.8*B9+1.2
11	=A10+1	=0.8*B10+1.2
12	=A11+1	=0.8*B11+1.2
13	=A12+1	=0.8*B12+1.2
14	=A13+1	=0.8*B13+1.2
15	=A14+1	=0.8*B14+1.2
16	=A15+1	=0.8*B15+1.2
17	=A16+1	=0.8*B16+1.2
18	=A17+1	=0.8*B17+1.2
19	=A18+1	=0.8*B18+1.2
20	=A19+1	=0.8*B19+1.2
21	=A20+1	=0.8*B20+1.2
22	=A21+1	=0.8*B21+1.2

	A	B
1	Worked Example 14	
2	n	t_n
3	1	0.20
4	2	1.36
5	3	2.29
6	4	3.03
7	5	3.62
8	6	4.10
9	7	4.48
10	8	4.78
11	9	5.03
12	10	5.22
13	11	5.38
14	12	5.50
15	13	5.60
16	14	5.68
17	15	5.74
18	16	5.80
19	17	5.84
20	18	5.87
21	19	5.90
22	20	5.92

Example

Aaron obtains a loan of $2000 from a bank, with monthly interest of 0.50% (that is, 6% per annum) and monthly repayments of $60.

Using the difference equation $t_{n+1} = rt_n + d$ with an initial term $2000 and d being -60 (as he is reducing the loan)

$r = 1 + \frac{0.5}{100}$ which becomes 1.005
Note that we start the first month as 0 and again we only need to complete Row 3 and the Fill Down.
There are missing rows which were simply removed to save space. You do not need to do this.
The 37th Row shows -$26.44 which means that he only had to make a payment of $60 - $26.64 = $33.36 in the 37th month.
In summary, Steve repaid 36 × 60 + 33.64 = $2193.36 over the 37 months, which means he paid $193.36 in interest.

	A	B
1	Worked Example 15	
2	Month	Debt
3	0	2000
4	=A3+1	=1.005*B3-60
5	=A4+1	=1.005*B4-60
6	=A5+1	=1.005*B5-60
7	=A6+1	=1.005*B6-60
8	=A7+1	=1.005*B7-60
9	=A8+1	=1.005*B8-60
33	=A32+1	=1.005*B32-60
34	=A33+1	=1.005*B33-60
35	=A34+1	=1.005*B34-60
36	=A35+1	=1.005*B35-60
37	=A36+1	=1.005*B36-60
38	=A37+1	=1.005*B37-60
39	=A38+1	=1.005*B38-60
40	=A39+1	=1.005*B39-60

	A	B
1	Worked Example 15	
2	Month	Debt
3	0	$2,000.00
4	1	$1,950.00
5	2	$1,899.75
6	3	$1,849.25
7	4	$1,798.49
8	5	$1,747.49
9	6	$1,696.22
33	30	$386.00
34	31	$327.93
35	32	$269.57
36	33	$210.92
37	34	$151.97
38	35	$92.73
39	36	$33.19
40	37	-$26.64

Exercise 3

Q1. A couple wish to buy a house and plan to borrow $100,000 with monthly repayments of $800 and a monthly interest of 0.50%.

a) Construct a difference equation which may be used to calculate the size of the debt at the end of each month.

b) Construct a spreadsheet which calculates the size of the debt each month.

c) From your spreadsheet, find how many months are required to fully repay the debt and how much is repaid in the last payment.

d) Calculate how much it has cost the couple in total for the loan of the $100,000.

e) Show that this sequence is neither arithmetic or geometric.

Q2. Carol wishes to buy a new Porsche and plans to borrow $150 000 with monthly repayments of $1500 and a monthly interest of 0.60% charged on the amount in the account.

a) Construct a difference equation which may be used to calculate the size of the debt each month.

b) Construct a spreadsheet which calculates the size of the debt each month.

c) From your spreadsheet, find how many months are required to fully repay the debt and how much is repaid in the last payment and how much interest Carol paid.

Q3. Marion is injected with 60 mL of a pain killing drug. Every four hours, the amount of drug in her bloodstream is reduced by 40%. To compensate for this loss, she is given an extra 15 mL of the drug every four hours.

a) Find a recurrence relation for the amount of drug in Marion's bloodstream when she is given her *nth* dose.

b) How much of the drug is in her bloodstream 12 hours after the initial dose is administered (use a spreadsheet). Remember each term of this sequence represents a four hour difference.

c) Eventually the amount of pain killing drug in Marion's bloodstream stabilizes. In the long run how many mL of pain killer will be present in Marion's bloodstream?

Q4. A vegetable farmer uses water from a storage tank to water his crops. The tank initially contains 50000 litres. Each day, 10% of the water in the tank is used to water the crops and each evening an extra 3000 litres is pumped into the tank from a nearby river. This pattern continues each day.

a) Write a difference equation for this scenario.

b) Use a spreadsheet to determine on which morning the volume of water first falls below 40000 litres.

c) How many litres of water will be contained in the tank in the long term?

Q5.A hedge is planted around the border of a country property. Each year the hedge grows by an estimated 80 cm and at the end of the year it is trimmed to 60% of its total height. When the hedge is planted it is 50 cm tall. Once the hedge reaches a certain height, this height is maintained. What is this height?

Task 1

There is a famous legend about the origin of chess which goes like this. When the inventor of the game showed it to an emperor of India, the emperor was so impressed by the new game, that he said to the inventor

"Name your reward!"

The man responded,

"Oh emperor, my wishes are simple. I only wish for this. Give me one grain of wheat for the first square of the chessboard, two grains for the next square, four for the next, eight for the next and so on for all 64 squares, with each square having double the number of grains as the square before."

The emperor agreed, amazed that the man had asked for such a small reward – or so he thought. He asked for a bag of wheat to be brought and assumed the man would soon be on his way. The bag of wheat was soon used up and the emperor now realised that he could never grant the reward as from then one, he would be doubling the bags of wheat on each square.

1. How many grains of wheat are needed for the 64^{th} square?
2. How many grains of wheat are needed altogether?
3. If a grain of wheat weighs 0.1 grams, how many kilograms of wheat does this represent? (1000 grams = 1 kilogram)
4. Compare this to the world's total production of wheat for 1 year.

Show how the answers are calculated (show your working out) but you may need to look up the answers as the numbers are very big. Express your final answers in scientific notation where necessary.

Task 2

The area covered by an ice shelf was measured over several years. At the beginning of the first year the ice shelf was spread over an area of 1437 square kilometres, and it was found that this area decreased by 2% each year over the recording period.

a) What area was covered by the ice shelf after 13 years of recording? Give your answer to the nearest square kilometre.

b) By what percentage did the ice shelf decrease over the 13 years.

Marking Rubric **Week 11** **Name:**

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work, including exercises and Mathspace task, of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation Task	Student completes the investigation task of the week to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and Communications	Student responses are accurate and appropriate in presentation of mathematical ideas, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and Techniques	Student submitted work selects and applies appropriate mathematical techniques to solve practical problems and demonstrates proficiency in the use of mathematical facts, techniques, and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises, Mathspace/online task, and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
				FINAL	**/20**

Student Reflection: How did you go with this week's work? What did you learn? What did you find easy? What do you need to work on?