Week 7/8
Term 1
2021

HAWKER COLLEGE

Engage | Inspire | Achieve

Learning Brief

MA3

Goals

This fortnight:

- use recursion to generate a geometric sequence
- display the terms of a geometric sequence in both tabular and graphical form and demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations
- deduce a rule for the *n*th term of a particular geometric sequence from the pattern of the terms in the sequence, and use this rule to make predictions
- use geometric sequences to model and analyse (numerically, or graphically only) practical problems involving geometric growth and decay; for example, analysing a compound interest loan or investment, the growth of a bacterial population that doubles in size each hour, the decreasing height of the bounce of a ball at each bounce; or calculating the value of office furniture at the end of each year using the declining (reducing) balance method to depreciate.

Theoretical Components

Resources:
For this week the theory work is in the *PDF file:* Week *7* and 8 Notes & Exercises

Sum to infinity and the concept of convergence

https://www.youtube.com/watch?v=PSA6mr0Lzk

Sum of a geometric series:

$$S_n = \frac{a(r^n-1)}{r-1};\ r \neq 1$$

Sum to infinity:

$$S_\infty = \frac{a}{1-r}$$

Knowledge Checklist

- Adding terms of a geometric series
- Simultaneous equations
- For *r* between -1 and 1 as *n* gets bigger, r^n gets smaller
- Adding an infinite number of terms when -1 < r < 1

Practical Components

There are questions to be answered in the booklet *Week 7and 8 Notes & Exercises*

All mathspace tasks, booklets, and investigations should be completed and submitted by the time of the Test.

Spend time in Week 8 week to catch up on any work you have yet to compete.

You should ensure that you have all investigations neatly presented and marked. Remember that your investigations from this term is worth 20% of your grade.

Investigation

Prepare a 'summary sheet' that will be useful to you in the test. You can have two sides of an A4 page. This **MUST** be handwritten.
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Quiz

Mathspace Task – Arithmetic and Geometric Sequences

https://mathspace.co/student/tasks/TopicCustomTask-457569/

Knowledge Checklist

Week 1

- Meaning of bivariate
- Independent vs dependent data
- Two-way frequency table

Week 2

- Parallel dot plots
- Divided bar charts

Week 3

- Scatter plots
- q-correlation coefficient
- Pearson’s correlation coefficient
- Coefficient of determination

Week 4

- Fitting straight lines to bivariate data
- Least squares regression line
- Interpolation and extrapolation

Week 5

- Sequences
- Arithmetic sequences
- Using simultaneous equation

Week 6

- Arithmetic series
- Geometric sequences

Week 7

- Geometric series
- Sum to infinity

Geometric Series

When we add up or sum the terms in a sequence, we get the series for that sequence. If we look at the geometric sequence {2, 6, 18, 54, …} where the first term $a = 2$ and the common ratio is 3, the series becomes $2 + 6 + 18 + 54 + \cdots$

$$S_1 = t_1 = 2$$
$$S_2 = t_1 + t_2 = 2 + 6 = 8$$
$$S_3 = t_1 + t_2 + t_3 = 2 + 6 + 18 = 26$$
$$S_4 = t_1 + t_2 + t_3 + t_4 = 2 + 6 + 18 + 54 = 80$$

The sum of the first n terms of a geometric sequence is given by:

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1}$$

Where a is the first term of the sequence and r is the common ratio.

Example

Find the sum of the first 9 terms of the sequence 0.25, 0.5, 1, 2, 4, …

Solution

The series is geometric and $a = 0.25$

Find the value of r by testing ratios of the given terms.

$$\frac{t_2}{t_1} = \frac{0.5}{0.25} = 2$$
$$\frac{t_3}{t_2} = \frac{1}{0.5} = 2$$
$$\frac{t_4}{t_3} = \frac{2}{1} = 2$$
$$\frac{t_5}{t_4} = \frac{4}{2} = 2$$

Using $S_n = \frac{a(r^n - 1)}{r - 1}$ gives

$$S_9 = \frac{0.25(2^9 - 1)}{2 - 1}$$
$$S_9 = \frac{0.25(512 - 1)}{1}$$
$$S_9 = 127.75$$

Example

The third term of a geometric sequence is 11.25 and the sixth term is 303.75. Find the sum of the first 10 terms of the sequence correct to 1 decimal place.

Solution

We need to find a and r. We know that $t_3 = 11.25$ and that $t_n = ar^{n-1}$.	$t_3 = ar^2$ $t_3 = 11.25$ $ar^2 = 11.25$ Equation 1
We know that $t_6 = 303.75$ and that $t_n = ar^{n-1}$.	$t_6 = ar^5$ $t_6 = 303.75$ $ar^5 = 303.75$ Equation 2
Solve the two equations simultaneously by eliminating *a*, to find *r*. Equation 2 divided by equation 1	$\frac{ar^5}{ar^2} = \frac{303.75}{11.25}$ $r^3 = 27$ $r = 3$
To find *a*, substitute the value of *r* into equation 1	$a \times 3^2 = 11.25$ $a = 1.25$
Since $r > 1$, use $S_n = \frac{a(r^n-1)}{r-1}$	$S_{10} = \frac{1.25(3^{10}-1)}{3-1}$ $S_{10} = 36905$
Write your answer	The sum of the first ten terms of the geometric series is 36905

Exercise 1

1. Find the sum of the following:

    a. First 12 terms of the geometric sequence 2, 6, 18, 54, 162, …

    b. First 7 terms of the geometric sequence 5, 35, 245, 1715, 12 005, …

c. First 7 terms of the geometric sequence 5, 35, 245, 1715, 12 005, …

d. First 15 terms of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

e. First 12 terms of the geometric sequence -0.1, -0.4, -1.6, -6.4, -25.6, …

f. First 11 terms of the geometric sequence 128, 64, 32, 16, 8, …

2. The second term of a geometric sequence is 10 and the fifth is 80. Find the sum of the first 12 terms of the sequence.

3. The second term of a geometric sequence is 6 and the fifth term is 48. Find the sum of the first 15 terms of the sequence.

4. How many terms of the geometric sequence 3, 6, 12, 24, 48, … are required for the sum to be greater than 3000? Use trial and error.

5. $t_{n+1} = 2t_n, t_1 = \frac{3}{2}$. Find S_{10}

6. On the first day, Abbey hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.
    a. Write the geometric sequence for the first five days of the above real-life situation.

    b. Find the value of r.

    c. How many people are told of the rumour on the 12^{th} day?

    d. How many people altogether have heard the rumour on the 12^{th} day?

The infinite sum of a geometric sequence where $r < 1$

A student stands at one side of a road, 10 metres wide, and walks half-way across. The student then walks half of the remaining distance across the road, then half the remaining distance again and so on. Will the student ever make it *past* the other side of the road? And does the width of the road affect your answer?

Here, we have a sequence $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \frac{5}{32}, \ldots$

The terms are getting smaller and smaller, in fact, no matter how many terms we add the sum cannot be greater than 10 (given that the road is only 10 m wide).

This type of scenario is called a 'sum to infinity'.

Consider the following sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$ Here $r = \frac{1}{2}$

As n increases the value of the term decreases. $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}, \left(\frac{1}{2}\right)^{20} = \frac{1}{1048576}$

As n gets larger, the term gets smaller. In fact, as n approaches infinity, the term approaches zero.

As $n \to \infty$, $r^n \to 0$ Note: This only occurs when r is a fraction between -1 and 1.
This is written as $-1 < r < 1$.

So, for very large values of n, our formula $S_n = \frac{a(1-r^n)}{1-r}$ becomes $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$.

The sum to infinity of a geometric sequence for which $-1 < r < 1$ is given by:

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

Example

Find the sum to infinity for the sequence $t_n: \{10,\ 1,\ 0.1, \ldots\}$

Solution

We know that the first term, a, is 10 and $r = 0.1$.	$a = 10, r = 0.1$
Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1-r}; \lvert r \rvert < 1$
Substitute $a = 10$ and $r = 0.1$ into the formula and evaluate.	$S_\infty = \frac{10}{1-0.1} = \frac{10}{0.9} = \frac{100}{9}$

Thus, no matter how many terms we add the sum can never exceed $\frac{100}{9}$.

Example

Find the fourth term in the geometric sequence whose first term is 6 and whose sum to infinity is 10.

Solution

Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1-r}; \; \lvert r \rvert < 1$
From the question, it is known that the infinite sum is equal to 10 and that the first term a is 6. Write down this information.	$a = 6; S_\infty = 10$
Substitute known values into the formula. Solve for r.	$10 = \frac{6}{1-r}$ $10(1-r) = 6$ $10 - 10r = 6$ $-10r = -4$ $r = 0.4$
Write the general formula for the nth term of the geometric sequence.	$t_n = ar^{n-1}$
To find the fourth term, substitute $a = 6, n = 4$ and $r = 0.4$ into the formula and evaluate.	$t_4 = 6 \times 0.4^{4-1} = 0.384$

Exercise 2

a. $t_n: \left\{1, \frac{1}{2}, \frac{1}{4}, \dots\right\}$

b. $t_n: \left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots\right\}$

c. $t_n: \left\{1, \frac{1}{3}, \frac{1}{9}, \dots\right\}$

d. $t_n: \left\{1, \frac{2}{3}, \frac{4}{9}, \dots\right\}$

2. For the infinite geometric sequence $\left\{\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots\right\}$. Find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

3. A sequence of numbers is defined by $t_n\!:\{9,\ -3,\ \ 1, \ldots\}$
   a. Find the sum of the first 9 terms.

   b. Find the sum to infinity, S_∞.

4. A sequence of numbers is defined by $t_n = 3 \times \left(\frac{1}{2}\right)^{n-1},\ n \in \{1,\ 2,\ 3, \ldots\}$.
   a. List the first 5 terms of the sequence.

   b. Find the sum of the first 20 terms.

   c. Find the sum to infinity, S_∞

2021 MA3

Investigation Week 7 and 8

Prepare a ‘summary sheet’ that will be useful to you in the test. You can have up to two sides of an A4 page. This **MUST** be handwritten.
Submit your summary sheet, with your name clearly written at the top, at the end of the test.
Your summary sheet may include rules and formulae, worked examples, diagrams, graphs, etc

Knowledge Checklist

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Week 7

- Geometric series
- Sum to infinity

Marking Rubric **Week 7/8** **Name:**

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work, including exercises and Mathspace task, of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation Task	Student completes the investigation task of the week to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and Communications	Student responses are accurate and appropriate in presentation of mathematical ideas, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and Techniques	Student submitted work selects and applies appropriate mathematical techniques to solve practical problems and demonstrates proficiency in the use of mathematical facts, techniques, and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises, Mathspace/online task, and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
				FINAL	**/20**

Student Reflection: How did you go with this week's work? What did you learn? What did you find easy?
What do you need to work on?