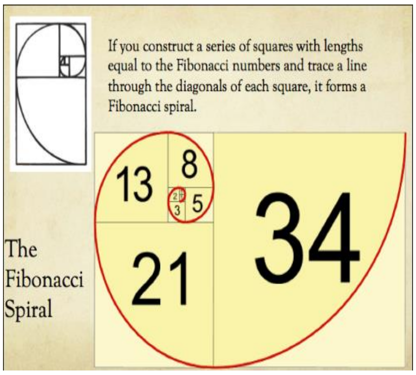
Week 5/6
Term 1
2021

HAWKER COLLEGE
Engage | Inspire | Achieve

Learning Brief
MA3



Goals

This fortnight:

- Use recursion to generate arithmetic and geometric sequences
- Display the terms of an arithmetic and geometric sequence in both tabular and graphical form and demonstrated that arithmetic sequences can be used to model linear growth and decay in discrete situations
- Deduce a rule for the *n*th term of a particular arithmetic and geometric sequence from the pattern of the terms in an arithmetic and geometric sequence, demonstrate that geometric sequences can be used to model exponential growth and decay in discrete situations
  and use this rule to make predictions

Theoretical Components

Resources:
For this week the theory work is in the *PDF file:* Week 5/6 Notes & Exercises

The following link provides additional information on sequences
https://www.mathsisfun.com/algebra/sequences-series.html

The fascinating world of Fibonacci numbers
https://www.youtube.com/watch?v=iEnR8zupK0A

Arithmetic sequence:
$$T_n = a + (n-1)d$$
What is a geometric sequence?
https://www.youtube.com/watch?v=1z8QKFFU3Hc

Sum of an arithmetic sequence:
$$S_n = \frac{n}{2}[2a + (n-1)d]$$

Geometric progressions:
$$t_n = ar^{n-1}$$

Practical Components

There are questions to be answered in the booklet *Week 5/6 Notes & Exercises*

Knowledge Checklist

- What is a sequence
- The three ways of defining a sequence
- Iterative
- Using a recursive formula
- Defining an arithmetic sequence
- General term
- Solving simultaneously
- Adding terms of an arithmetic sequence
- Arithmetic series
- Multiplying terms
- Geometric sequences

Investigation

See the end of the brief ☺

Quiz

Mathspace.co in-class quiz Monday Week 5

Remember to complete your reflection for this brief ☺

MATHEMATICAL APPLICATIONS 3

WEEK 5 and 6 NOTES & EXERCISES

Sequences and series

Number patterns

An important skill in mathematics is to be able to:

- Recognise patterns in sets of numbers,
- Describe the patterns in words, and
- Continue the patterns.

Sequences of numbers play an important part in our everyday life. For example, the following sequence: 2.25, 2.34, 2.58, 2.49, 2.65, … gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?

The following sequence is more predictable:

10 000, 9000, 8100, …

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. This compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted: during the second minute, 9000, and so on. Can you predict the next number in the sequence? You are correct if you said 7 290. Each successive term is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They may be finite in number or infinite. Number sequences may follow an easily recognisable pattern, or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting to name a few uses.

Sequences which follow a pattern can be described several different ways. They may be listed in sequential order, they may be described as a functional definition, or they may be described in an iterative definition.

1. Listing in sequential order

3, 7, 11, 15, … forms a number sequence. The first term is 3, second term is 7, third term is 11, etc.
We can describe this pattern in words:
"The sequence starts at 3 and each term is 4 more than the previous one."
Thus, the fifth term is 19, and the sixth term is 23, etc.

Example

Describe the sequence: 14, 17, 20, 23, … and write down the next two terms.

Solution

The sequence starts at 14 and each term is 3 more than the previous term. The next two terms are 26 and 29.

Exercise 1

1. Write down the first four terms of the sequence if you start with:

   a. 4 and add 9 each time

   b. 45 and subtract 6 each time

   c. 2 and multiply by 3 each time

   d. 96 and divide by 2 each time

2. Describe the following number patterns and write down the next 3 terms:

   a. 1, 4, 9, 16, …

   b. 1, 8, 27, 64, …

2. Functional definition

A functional definition is expressed in the form: $t_n = 2n - 7, n \in \{1,2,3,4,\ldots\}$. Using this definition, the *n*th term can be readily calculated. For this example: $t_1 = 2 \times 1 - 7 = -5$, $t_2 = 2 \times 2 - 7 = -3$, $t_3 = 2 \times 3 - 7 = -1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value n = 100 into the expression for t_n.

Example

Find the first four terms of the sequence: $d_n = 4.9n^2, n \in \{1,2,3,\ldots\}$

Solution

$d_1 = 4.9 \times 1^2 = 4.9$, $d_2 = 4.9 \times 2^2 = 19.6$, $d_3 = 4.9 \times 3^2 = 44.1$ and $d_4 = 4.9 \times 4^2 = 78.4$. The sequence is $\{4.9, 19.6, 44.1, 78.4, \ldots\}$

Exercise 2

Find the first, fifth and tenth terms in the following sequences

a. $t_n = 2n - 5, n \in \{1,2,3, \dots\}$

b. $t_n = \frac{n}{n+1}, n \in \{1,2,3, \dots\}$

c. $t_n = (-1)^n + n, n \in \{1,2,3, \dots\}$

d. $t_n = n^2 - n + 41, n \in \{1,2,3, \dots\}$

3. Iterative definition

An iterative definition is expressed in the form: $t_{n+1} = 3t_n - 2, t_1 = 6$. This definition looks complicated, but it is actually straight forward. The word iteration means the calculation of the next term from the previous ter, using the same procedure. The symbol t_{n+1} simply means the next term after the term t_n.

Example

Find the first four terms of the sequence above: $t_{n+1} = 3t_n - 2, t_1 = 6$

Solution

The first term, t_1, is 6 (this is given in the definition).
The second term, t_2, is $3 \times 6 - 2 = 16$
The third term, t_3, is $3 \times 16 - 2 = 46$
The fourth term, t_4, is $3 \times 46 - 2 = 136$

Exercise 3

1. Find the first 4 terms in the following sequences.

   a. $t_{n+1} = t_n + 2, t_1 = 3$

   b. $t_{n+1} = 3t_n, t_1 = 2$

   c. $t_{n+1} = t_n - 7, t_1 = 14$

   d. $t_{n+2} = t_{n+1} + t_n, t_1 = 1, t_2 = 1$

Arithmetic Sequences

An arithmetic sequence is a sequence where there is a common different between any two successive terms.

For example: 2, 5, 8, 11, 14, … is arithmetic as $5 - 2 = 8 - 5 = 11 - 8 = 14 - 11$, etc, because they each have a difference of 3.
Likewise, 31, 27, 23, 19, … is arithmetic as $31 - 27 = 27 - 23 = 23 - 19$, etc, because they each have a difference of -4.

Algebraically, $\boldsymbol{T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \cdots = T_{n+1} - T_n}$

Why 'arithmetic'

If a, b, and c are any three consecutive terms of an arithmetic sequence then $b - a = c - b$ (equating common differences). Therefore, through rearranging $2b = a + c$, which gives $b = \frac{a+c}{2}$. The middle term is the arithmetic mean (average) of the terms on each side of it. Hence the name arithmetic sequence.

The General Term Formula

In an arithmetic sequence, the first term is denoted by a and the common difference by d. The position of the term is denoted by n, such as for the first term $n - 1$, second term $n - 2$, third term $n - 3$, etc.

Consider the sequence 2, 9, 16, 23, 30, … For this sequence, $a = 2$ and $d = 7$

2	9	16	23
a	$a + d$	$a + d + d$	$a + d + d + d$
a	$a + d$	$a + 2d$	$a + 3d$

Thus, for a given position, n, the term is given by $a + (n - 1)d$

Therefore, the general term of an arithmetic progression is given by:

$$\boldsymbol{T_n = a + (n - 1)d}$$

For the sequence above, the formula for its general term is given by $T_n = 2 + (n - 1) \times 7$
So, the 50th term is $2 + (50 - 1) \times 7 = 2 + 49 \times 7 = 2 + 343 = 345$

Exercise 4

1. Consider the sequence 6, 17, 28, 39, 50, …
   a. Show that the sequence is arithmetic (Check for a common difference).

   b. Find the formula for its general term.

   c. Find the 50th term.

   d. Is 325 a member of the sequence?

2. Consider the sequence 87, 83, 79, 75, …

   a. Show that the sequence is arithmetic.

   b. Find the formula for the general term.

c. Find the 40^{th} term.

d. Is -143 a member of the sequence?

3. Show that the following sequences are arithmetic:

a. {-0.12, 3.48, 7.08, …}

b. $\left\{\frac{5}{9}, -\frac{1}{9}, -\frac{7}{9}, \ldots\right\}$

c. $\left\{5\frac{2}{3}, 7\frac{4}{15}, 8\frac{13}{15}, \ldots\right\}$

d. $\{x + 9, 2x + 7, 3x + 5, \ldots\}$

4. For the arithmetic sequence {22, m, n, 37, …}, find the values for m and n.

Example

Find k given that 3k+1, k and -3 are consecutive terms of an arithmetic sequence.

Solution

Since the terms are consecutive, $\mathrm{k} - (3\mathrm{k} + 1) = -3 - \mathrm{k}$ (2^{nd} term – 1^{st} term = 3^{rd} term – 2^{nd} term).
$\mathrm{k} - 3\mathrm{k} - 1 = -3 - \mathrm{k}$ (removing of brackets)
$-2\mathrm{k} - 1 = -3 - \mathrm{k}$ (collecting like terms)
$-1 + 3 = 2\mathrm{k} - \mathrm{k}$ (rearranging)
Therefore, $\mathrm{k} = 2$

5. Find k given the consecutive arithmetic terms:

   a. 32, k, 3

   b. $\mathrm{k}+1, 2\mathrm{k}+1, 13$

Example

Find the general term T_n for an arithmetic sequence with $T_3 = 8$ and $T_8 = -17$

Solution

$T_3 = 8$ which tells us, $a + 2d = 8 \ldots$ (1) using $T_n = a + (n-1)d$
$T_8 = -17$ which tells us, $a + 7d = -17 \ldots$ (2)
We now solve (1) and (2) simultaneously

$a + 2d = 8$
$a + 7d = -17$
(1)-(2) gives $-5d = 25$
Therefore $d = -5$

Substitute $d = -5$ into (1) gives
$a + 2(-5) = 8$
$a - 10 = 8$
$a = 18$

Therefore, the sequence is 18, 13, 8, 3, -2, etc and the general equations is $T_n = 18 + (n-1) \times -5$

6. Find the general term T_n for an arithmetic sequence given that:
   a. $T_7 = 41$ and $T_{13} = 77$

   b. The seventh term is 1 and fifteenth term is -39

Sum of a given number of terms of an arithmetic sequence

When the terms of an arithmetic sequence are added together, an arithmetic series is formed.
So, 5, 9, 13, 17, 21, … is an arithmetic sequence,
Whereas 5 + 9 + 13 + 17 + 21 + … is an arithmetic series.

The sum of *n* terms of an arithmetic sequence is given by S_n.
The formula for the sum of *n* terms of an arithmetic sequence when the value of *a* and *d* are known is given by;

$$\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$$

Example

Find the sum of 4 + 7 + 10 + 13 + … to 50 terms.

Solution

The series is arithmetic will $a = 4, d = 3$ and $n = 50$.
Using $S_n = \frac{n}{2}[2a + (n-1)d]$ gives

$$S_n = \frac{50}{2}[2 \times 4 + (50-1) \times 3]$$
$$S_n = 25[8 + 49 \times 3]$$
$$S_n = 3875$$

Exercise 5

1. Find the sum of the following:

   a. 3 + 7 + 11 + 15 + … to 20 terms

   b. 100 + 93 + 86 + 79 + … to 40 terms

   c. $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + \ldots$ to 50 terms

2. Find the sum of the first 100 positive integers

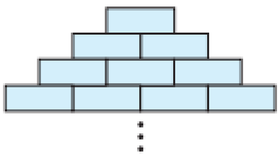
3. A sequence is 5, 7, 9, 11, …, how many consecutive terms need to be added to obtain 357?

4. The first term in an arithmetic sequence is 5 and the sum of the first 20 terms is 1240. Find the common difference, d.

5. The first term is 50 and the 10th term is -40. Find S_{10}

6. Find the sum of 5 + 8 + 11 + 14 + … + 101.
   Note: First use $T_n = a + (n - 1)d$ to find the value of *n*.

7. A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers are placed?



Geometric Sequences

A farmer is breeding worms which he hopes to sell to the local shire councils for use in the decomposition of waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $1 + \frac{1}{10}$ or 1.1. He starts off with 10 kg of worms. By the beginning of the second week, he will expect $10 \times 1.1 = 11$kg of worms, by the start of the third week, he would expect $11 \times 1.1 = 10 \times 1.1^2 = 12.1$kg of worms, and so on. This is an example of a geometric sequence.

A geometric sequence is one whereby the first term is multiplied by a number, known as the common ratio, to create the second term which is multiplied by the common ratio to create the third, and so on. The first term in a geometric sequence is referred to as *a* and the common ration is referred to as *r*. consider the geometric sequence where $a = 1$ and $r = 3$. The terms in the sequence are: 1, 3, 9, 27, 81, …

To discover the common ratio, *r*, of a geometric sequence you need to calculate the ratio of successive terms, namely $\frac{t_2}{t_1}$. Alternatively, you could calculate $\frac{t_3}{t_2}$ or $\frac{t_4}{t_3}$.

Exercise 6

1. Which of the following are geometric sequences?
2.

a. 1, 2, 4, 8, 16, … b. 2, 6, 18, 54, 162, … c. 1, 4, 16, 64, 256, …

d. 2, 6, 12, 24, 48, … e. 1, 5, 25, 100, 125, … f. 0, 2, 4, 8, 16, …

g. -2, 4, -8, 16, -32, … h. -1, -5, 10, -20, -40, … i. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

3. In question 1, for the geometric sequences you have identified, find a and r.

a. b. c.

d. e. f.

Find the terms of a geometric sequence

Consider the finite geometric sequence of five terms for which $a = 3$ and $r = 4$.

3, 12, 48, 192, 768.

Now:

$t_1 = 3$	$t_1 = a$	
$t_2 = 3 \times 4$	$t_2 = a \times r$	$t_2 = a \times r^1$
$t_3 = 3 \times 4 \times 4$	$t_3 = a \times r \times r$	$t_3 = a \times r^2$
$t_4 = 3 \times 4 \times 4 \times 4$	$t_4 = a \times r \times r \times r$	$t_4 = a \times r^3$
$t_5 = 3 \times 4 \times 4 \times 4 \times 4$	$t_5 = a \times r \times r \times r \times r$	$t_5 = a \times r^4$

We should notice a pattern emerging. That pattern can be described by the equation:

$$t_n = 3 \times 4^{n-1}$$

For example, if $n = 5$, then $t_5 = 3 \times 4^4$

We can generalise this rule for all geometric sequences.

$$\boldsymbol{t_n = ar^{n-1}}$$

Where t_n is the *n*th term,
a is the first term,
r is the common ratio.

This rule enables us to find any term of a geometric sequence provided we know the value of *a* and *r*.

Example

Find the 12th term of the geometric sequence: 2, 10, 50, 250, 1250, …

Solution

Step 1: Find the value for *a*. $a = 2$
Step 2: Find the value for *r* (if stated it is a geometric sequence). $r = \frac{10}{2} = 5$
Step 3: Use the rule to find the 12th term. $t_{12} = 2 \times 5^{11} = 97{,}656{,}250$
Step 4: Write your answer. The 12th term is 97 656 250

Exercise 7

1. Find the value of the term specified for the given geometric sequences.

    a. Find the 15th term of the geometric sequence 2, 8, 32,128, 512, …

    b. Find the 10th term of the geometric sequence 2, 12, 72, 432, 2592, …

c. Find the 20th term of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

d. Find the 8th term of the geometric sequence 3.1, 8.06, 20.956, 54.4856, 141.66256, …

e. Find the 9th term of the geometric sequence -2, -8, -32, -128, -512, …

f. Find the 10th term of the geometric sequence $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}$, ...

Example

The 2nd term of a geometric sequence is 8 and the 5th term is 512. Find the 10th term of this sequence.

Solution

Step 1: We know that $t_2 = 8$ and that $t_n = ar^{n-1}$.	$t_2 = ar^1$ $t_2 = 8$	
	$ar^1 = 8$	Equation 1
Step 2: We know that $t_5 = 512$ and that $t_n = ar^{n-1}$.	$t_5 = ar^4$ $t_5 = 512$	
	$ar^4 = 512$	Equation 2
Step 3: Solve the two equations simultaneously by eliminating *a*, to find *r*.	$\frac{ar^4}{ar^1} = \frac{512}{8}$	
Divide equation 2 by equation 1	$r^3 = 64$ $r = 4$	
Step 4: To find *a*, substitute the value of *r* into equation 1	$a \times 4 = 8$ $a = 2$	
Step 5: Write down the rule.	$t_n = 2 \times 4^{n-1}$	
Step 6: Find the 10th term, let $n = 10$	$t_{10} = 2 \times 4^9$ $t_{10} = 524{,}288$	
Step 7: Write your answer	The 10th term in the sequence is 524 288	

2. Find the value of the term specified for the specified geometric sequences.
   a. The 2nd term of a geometric sequence is 6 and the 5th term is 162. Find the 10th term.

   b. The 2nd term of a geometric sequence is 6 and the 5th term is 48. Find the 12th term.

3. For the geometric sequence 3, *p*, *q*, 192, … find the values for *p* and q.

4. The first three terms of a geometric sequence are 2, 6, and 18. Which numbered term would be the first to exceed 1 000 000 in this sequence? You will need to use trial and error to solve this.

The Fibonacci and Lucas Sequences

Leonardo Fibonacci of Pisa was a mathematician in the 12^{th} century, Italy. He discovered a number series from which one can derive the Golden Mean by charting the population of rabbits. Here is the beginning of the sequence:

1, 1, 2, 3, 5, 8,13, 21, 34, …

The Lucas numbers or Lucas series are an integer sequence named after the mathematician François Édouard Anatole Lucas (1842-1891). The sequence named after him is closely related to the Fibonacci sequence.

These sequences are defined recursively by:

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}, \text{for } n \geq 3$$

$$L_1 = 1, L_2 = 3, L_n = L_{n-1} + L_{n-2}, \text{for } n \geq 3$$

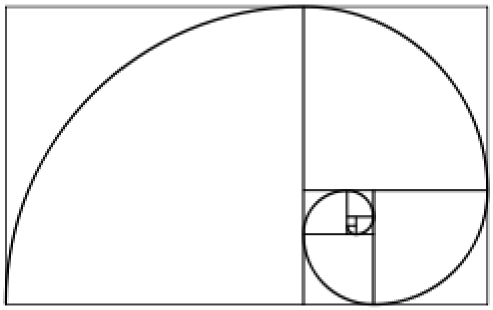
Each number is the sum of the two proceeding numbers.

1. Write out the first 12 terms of each sequence.

2. Explain why every third term of each sequence is even and the rest are odd.

3. Write out the sequence $L_1 + F_1, L_2 + F_2, L_3 + F_3, \ldots$ and $L_1 - F_1, L_2 - F_2, L_3 - F_3, \ldots$

4. How do the two sequences relate to the Fibonacci sequence?



Marking Rubric **Week 5/6** **Name:**

CRITERIA	EXPECTATIONS	POSS	MULT	GIVEN	TOTAL
Practical	Student completes practical work, including exercises and Mathspace task, of the brief to an acceptable standard set by the teacher.	**2**	**3**		**/6**
Investigation Task	Student completes the investigation task of the week to an acceptable standard set by the teacher.	**2**	**2**		**/4**
Reasoning and Communications	Student responses are accurate and appropriate in presentation of mathematical ideas, with clear and logical working out shown.	**4**	**-**		**/4**
Concepts and Techniques	Student submitted work selects and applies appropriate mathematical techniques to solve practical problems and demonstrates proficiency in the use of mathematical facts, techniques, and formulae.	**4**	**-**		**/4**
	Submission Guidelines				
Timeliness	Student submits the exercises, Mathspace/online task, and investigation by the set deadline. See scoring guidelines for specific details.	**2**	**-**		**/2**
				FINAL	**/20**

Student Reflection: How did you go with this week's work? What did you learn? What did you find easy?
What do you need to work on?