MATHEMATICAL APPLICATIONS 3

WEEK 7 NOTES & EXERCISES

Geometric Series

When we add up or sum the terms in a sequence, we get the series for that sequence. If we look at the geometric sequence {2, 6, 18, 54, …} where the first term $a = 2$ and the common ratio is 3, the series becomes $2 + 6 + 18 + 54 + \cdots$

$$S_1 = t_1 = 2$$
$$S_2 = t_1 + t_2 = 2 + 6 = 8$$
$$S_3 = t_1 + t_2 + t_3 = 2 + 6 + 18 = 26$$
$$S_4 = t_1 + t_2 + t_3 + t_4 = 2 + 6 + 18 + 54 = 80$$

The sum of the first n terms of a geometric sequence is given by:

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r - 1}; r \neq 1}$$

Where a is the first term of the sequence and r is the common ratio.

Example

Find the sum of the first 9 terms of the sequence 0.25, 0.5, 1, 2, 4, …

Solution

The series is geometric and $a = 0.25$

Find the value of r by testing ratios of the given terms.

$$\frac{t_2}{t_1} = \frac{0.5}{0.25} = 2$$
$$\frac{t_3}{t_2} = \frac{1}{0.5} = 2$$
$$\frac{t_4}{t_3} = \frac{2}{1} = 2$$
$$\frac{t_5}{t_4} = \frac{4}{2} = 2$$

Using $S_n = \frac{a(r^n - 1)}{r - 1}$ gives

$$S_9 = \frac{0.25(2^9 - 1)}{2 - 1}$$
$$S_9 = \frac{0.25(512 - 1)}{1}$$
$$S_9 = 127.75$$

Example

The third term of a geometric sequence is 11.25 and the sixth term is 303.75. Find the sum of the first 10 terms of the sequence correct to 1 decimal place.

Solution

We need to find a and r. We know that $t_3 = 11.25$ and that $t_n = ar^{n-1}$.	$t_3 = ar^2$ $t_3 = 11.25$	
	$ar^2 = 11.25$	Equation 1
We know that $t_6 = 303.75$ and that $t_n = ar^{n-1}$.	$t_6 = ar^5$ $t_6 = 303.75$	
	$ar^5 = 303.75$	Equation 2
Solve the two equations simultaneously by eliminating *a*, to find *r*.	$\frac{ar^5}{ar^2} = \frac{303.75}{11.25}$	
Equation 2 divided by equation 1	$r^3 = 27$ $r = 3$	
To find *a*, substitute the value of *r* into equation 1	$a \times 3^2 = 11.25$ $a = 1.25$	
Since $r > 1$, use $S_n = \frac{a(r^n-1)}{r-1}$	$S_{10} = \frac{1.25(3^{10}-1)}{3-1}$ $S_{10} = 36905$	
Write your answer	The sum of the first ten terms of the geometric series is 36905	

Exercise Set 1

1. Find the sum of the following
    a. First 12 terms of the geometric sequence 2, 6, 18, 54, 162, …

    b. First 7 terms of the geometric sequence 5, 35, 245, 1715, 12 005, …

c. First 15 terms of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

d. First 12 terms of the geometric sequence -0.1, -0.4, -1.6, -6.4, -25.6, …

e. First 11 terms of the geometric sequence 128, 64, 32, 16, 8, …

2. The second term of a geometric sequence is 10 and the fifth is 80. Find the sum of the first 12 terms of the sequence.

3. The second term of a geometric sequence is 6 and the fifth term is 48. Find the sum of the first 15 terms of the sequence.

4. How many terms of the geometric sequence 3, 6, 12, 24, 48, … are required for the sum to be greater than 3000? Use trial and error.

5. $t_{n+1} = 2t_n, t_1 = \frac{3}{2}$. Find S_{10}

6. On the first day, Abbey hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.
    a. Write the geometric sequence for the first five days of the above real-life situation.

    b. Find the value of r.

    c. How many people are told of the rumour on the 12th day?

    d. How many people altogether have heard the rumour on the 12th day?

The infinite sum of a geometric sequence where $r < 1$

A student stands at one side of a road, 10 metres wide, and walks half-way across. The student then walks half of the remaining distance across the road, then half the remaining distance again and so on. Will the student ever make it *past* the other side of the road? And does the width of the road affect your answer?

Here, we have a sequence $5, \frac{5}{2}, \frac{5}{4}, \frac{5}{8}, \frac{5}{16}, \frac{5}{32}, \ldots$

The terms are getting smaller and smaller, in fact, no matter how many terms we add the sum cannot be greater than 10 (given that the road is only 10 m wide).

This type of scenario is called a 'sum to infinity'.

Consider the following sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$ Here $r = \frac{1}{2}$

As n increases the value of the term decreases. $\left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$, $\left(\frac{1}{2}\right)^{20} = \frac{1}{1048576}$

As n gets larger, the term gets smaller. In fact, as n approaches infinity, the term approaches zero.

As $n \to \infty$, $r^n \to 0$ Note: This only occurs when r is a fraction between -1 and 1.
This is written as $-1 < r < 1$.

So, for very large values of n, our formula $S_n = \frac{a(1-r^n)}{1-r}$ becomes $S_n = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$.

The sum to infinity of a geometric sequence for which $-1 < r < 1$ is given by:

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

Example

Find the sum to infinity for the sequence t_n: $\{10,\ 1,\ 0.1, \dots\}$

Solution

We know that the first term, a, is 10 and $r = 0.1$.	$a = 10, r = 0.1$
Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1-r}; \lvert r \rvert < 1$
Substitute $a = 10$ and $r = 0.1$ into the formula and evaluate.	$S_\infty = \frac{10}{1-0.1} = \frac{10}{0.9} = \frac{100}{9}$

Thus, no matter how many terms we add the sum can never exceed $\frac{100}{9}$.

Example

Find the fourth term in the geometric sequence whose first term is 6 and whose sum to infinity is 10.

Solution

Write the formula for the sum to infinity.	$S_\infty = \frac{a}{1-r}; \lvert r \rvert < 1$
From the question, it is known that the infinite sum is equal to 10 and that the first term a is 6. Write down this information.	$a = 6; S_\infty = 10$
Substitute known values into the formula. Solve for r.	$10 = \frac{6}{1-r}$ $10(1-r) = 6$ $10 - 10r = 6$ $-10r = -4$ $r = 0.4$
Write the general formula for the nth term of the geometric sequence.	$t_n = ar^{n-1}$
To find the fourth term, substitute $a = 6, n = 4$ and $r = 0.4$ into the formula and evaluate.	$t_4 = 6 \times 0.4^{4-1} = 0.384$

Exercise Set 2

1. Find the sum to infinity of the following sequences

   a. $t_n\colon\left\{1,\ \frac{1}{2},\ \frac{1}{4},\ldots\right\}$

   b. $t_n\colon\left\{1,-\frac{1}{2},\frac{1}{4},-\frac{1}{8},\ldots\right\}$

   c. $t_n\colon\left\{1,\ \frac{1}{3},\ \frac{1}{9},\ldots\right\}$

   d. $t_n\colon\left\{1,\ \frac{2}{3},\ \frac{4}{9},\ldots\right\}$

2. For the infinite geometric sequence $\left\{\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots\right\}$. Find the sum to infinity. Consequently, find what proportion each of the first three terms contributes to this sum as a percentage.

3. A sequence of numbers is defined by $t_n\colon\{9,\ -3,\ \ 1,\ldots\}$

   a. Find the sum of the first 9 terms.

   b. Find the sum to infinity, S_∞.

4. A sequence of numbers is defined by $t_n = 3 \times \left(\frac{1}{2}\right)^{n-1}$, $n \in \{1,\ 2,\ 3, \ldots\}$.

   a. List the sequence.

   b. Find the sum of the first 20 terms.

   c. Find the sum to infinity, S_∞.