MATHEMATICAL APPLICATIONS 3

WEEK 6 NOTES & EXERCISES

Sum of a given number of terms of an arithmetic sequence

When the terms of an arithmetic sequence are added together, an arithmetic series is formed.
So, 5, 9, 13, 17, 21, … is an arithmetic sequence,
Whereas 5 + 9 + 13 + 17 + 21 + … is an arithmetic series.

The sum of *n* terms of an arithmetic sequence is given by S_n.
The formula for the sum of *n* terms of an arithmetic sequence when the value of *a* and *d* are known is given by;

$$\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$$

Example

Find the sum of 4 + 7 + 10 + 13 + … to 50 terms.

Solution

The series is arithmetic will $a = 4, d = 3$ and $n = 50$.
Using $S_n = \frac{n}{2}[2a + (n-1)d]$ gives

$$S_n = \frac{50}{2}[2 \times 4 + (50 - 1) \times 3]$$
$$S_n = 25[8 + 49 \times 3]$$
$$S_n = 3875$$

Exercise Set 1

1. Find the sum of the following
    a. 3 + 7 + 11 + 15 + … to 20 terms

    b. 100 + 93 + 86 + 79 + … to 40 terms

c. $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + \ldots$ to 50 terms

2. Find the sum of the first 100 positive integers

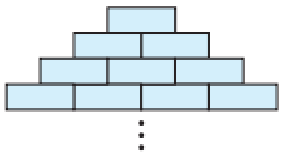
3. A sequence is 5, 7, 9, 11, …, how many consecutive terms need to be added to obtain 357?

4. The first term in an arithmetic sequence is 5 and the sum of the first 20 terms is 1240. Find the common difference, d.

5. The first term is 50 and the 10th term is -40. Find S_{10}

6. Find the sum of 5 + 8 + 11 + 14 + … + 101.
Note: First use $T_n = a + (n-1)d$ to find the value of *n*.

7. A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers are placed?



Geometric Sequences

A farmer is breeding worms which he hopes to sell to the local shire councils for use in the decomposition of waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $1 + \frac{1}{10}$ or 1.1. He starts off with 10 kg of worms. By the beginning of the second week, he will expect $10 \times 1.1 = 11$kg of worms, by the start of the third week, he would expect $11 \times 1.1 = 10 \times 1.1^2 = 12.1$kg of worms, and so on. This is an example of a geometric sequence.

A geometric sequence is one whereby the first term is multiplied by a number, known as the common ratio, to create the second term which is multiplied by the common ratio to create the third, and so on. The first term in a geometric sequence is referred to as *a* and the common ration is referred to as *r*. consider the geometric sequence where $a = 1$ and $r = 3$. The terms in the sequence are: 1, 3, 9, 27, 81, …

To discover the common ratio, *r*, of a geometric sequence you need to calculate the ratio of successive terms, namely $\frac{t_2}{t_1}$. Alternatively, you could calculate $\frac{t_3}{t_2}$ or $\frac{t_4}{t_3}$.

Exercise Set 2

1. Which of the following are geometric sequences?

a. 1, 2, 4, 8, 16, …

b. 2, 6, 18, 54, 162, …

c. 1, 4, 16, 64, 256, …

d. 2, 6, 12, 24, 48, …

e. 1, 5, 25, 100, 125, …

f. 0, 2, 4, 8, 16, …

g. -2, 4, -8, 16, -32, …

h. -1, -5, 10, -20, -40, …

i. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

2. In question 1, for the geometric sequences you have identified, find *a* and *r*.

a.

b.

c.

d.

e.

f.

g. h. i.

Find the terms of a geometric sequence

Consider the finite geometric sequence of five terms for which $a = 3$ and $r = 4$.

3, 12, 48, 192, 768.

Now:

$t_1 = 3$	$t_1 = a$	
$t_2 = 3 \times 4$	$t_2 = a \times r$	$t_2 = a \times r^1$
$t_3 = 3 \times 4 \times 4$	$t_3 = a \times r \times r$	$t_3 = a \times r^2$
$t_4 = 3 \times 4 \times 4 \times 4$	$t_4 = a \times r \times r \times r$	$t_4 = a \times r^3$
$t_5 = 3 \times 4 \times 4 \times 4 \times 4$	$t_5 = a \times r \times r \times r \times r$	$t_5 = a \times r^4$

We should notice a pattern emerging. That pattern can be described by the equation:

$$t_n = 3 \times 4^{n-1}$$

For example, if $n = 5$, then $t_5 = 3 \times 4^4$

We can generalise this rule for all geometric sequences.

$$\boldsymbol{t_n = ar^{n-1}}$$

Where t_n is the *n*th term,
a is the first term,
r is the common ratio.

This rule enables us to find any term of a geometric sequence provided we know the value of *a* and *r*.

Example

Find the 12th term of the geometric sequence: 2, 10, 50, 250, 1250, …

Solution

Step 1: Find the value for *a*.	$a = 2$
Step 2: Find the value for *r* (if stated it is a geometric sequence).	$r = \frac{10}{2} = 5$
Step 3: Use the rule to find the 12th term.	$t_{12} = 2 \times 5^{11} = 97{,}656{,}250$
Step 4: Write your answer.	The 12th term is 97 656 250

Exercise Set 3

1. Find the value of the term specified for the given geometric sequences.
   a. Find the 15th term of the geometric sequence 2, 8, 32,128, 512, …

b. Find the 10th term of the geometric sequence 2, 12, 72, 432, 2592, …

c. Find the 20th term of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, …

d. Find the 8th term of the geometric sequence 3.1, 8.06, 20.956, 54.4856, 141.66256, …

e. Find the 9th term of the geometric sequence -2, -8, -32, -128, -512, …

f. Find the 10th term of the geometric sequence $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

Example

The 2nd term of a geometric sequence is 8 and the 5th term is 512. Find the 10th term of this sequence.

Solution

Step 1: We know that $t_2 = 8$ and that $t_n = ar^{n-1}$.	$t_2 = ar^1$ $t_2 = 8$ $ar^1 = 8$	Equation 1
Step 2: We know that $t_5 = 512$ and that $t_n = ar^{n-1}$.	$t_5 = ar^4$ $t_5 = 512$ $ar^4 = 512$	Equation 2
Step 3: Solve the two equations simultaneously by eliminating *a*, to find *r*.	$\frac{ar^4}{ar^1} = \frac{512}{8}$	
Divide equation 2 by equation 1	$r^3 = 64$ $r = 4$	
Step 4: To find *a*, substitute the value of *r* into equation 1	$a \times 4 = 8$ $a = 2$	
Step 5: Write down the rule.	$t_n = 2 \times 4^{n-1}$	
Step 6: Find the 10th term, let $n = 10$	$t_{10} = 2 \times 4^9$ $t_{10} = 524{,}288$	
Step 7: Write your answer	The 10th term in the sequence is 524 288	

2. Find the value of the term specified for the specified geometric sequences.
    a. The 2nd term of a geometric sequence is 6 and the 5th term is 162. Find the 10th term.

    b. The 2nd term of a geometric sequence is 6 and the 5th term is 48. Find the 12th term.

3. For the geometric sequence 3, p, q, 192, … find the values for p and q

4. The first three terms of a geometric sequence are 2, 6, and 18. Which numbered term would be the first to exceed 1 000 000 in this sequence? You will need to use trial and error to solve this.