MATHEMATICAL APPLICATIONS 3

WEEK 5 NOTES & EXERCISES

Sequences and series

Number patterns

An important skill in mathematics is to be able to:

- Recognise patterns in sets of numbers,
- Describe the patterns in words, and
- Continue the patterns.

Sequences of numbers play an important part in our everyday life. For example, the following sequence: 2.25, 2.34, 2.58, 2.49, 2.65, … gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?

The following sequence is more predictable:

10 000, 9000, 8100, …

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. This compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted; during the second minute, 9000, and so on. Can you predict the next number in the sequence? You are correct if you said 7 290. Each successive term is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They may be finite in number or infinite. Number sequences may follow an easily recognisable pattern, or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting to name a few uses.

Sequences which follow a pattern can be described several different ways. They may be listed in sequential order, they may be described as a functional definition, or they may be described in an iterative definition.

1. Listing in sequential order

3, 7, 11, 15, … forms a number sequence. The first term is 3, second term is 7, third term is 11, etc.
We can describe this pattern in words:
"The sequence starts at 3 and each term is 4 more than the previous one."
Thus, the fifth term is 19, and the sixth term is 23, etc.

Example

Describe the sequence: 14, 17, 20, 23, … and write down the next two terms.

Solution

The sequence starts at 14 and each term is 3 more than the previous term. The next two terms are 26 and 29.

Exercise Set 1

1. Write down the first four terms of the sequence if you start with:
   a. 4 and add 9 each time
   b. 45 and subtract 6 each time
   c. 2 and multiply by 3 each time
   d. 96 and divide by 2 each time

2. Describe the following number patterns and write down the next 3 terms:
   a. 1, 4, 9, 16, …
   b. 1, 8, 27, 64, …

2. Functional definition

A functional definition is expressed in the form: $t_n = 2n - 7, n \in \{1,2,3,4,\dots\}$. Using this definition, the *n*th term can be readily calculated. For this example: $t_1 = 2 \times 1 - 7 = -5$, $t_2 = 2 \times 2 - 7 = -3$, $t_3 = 2 \times 3 - 7 = -1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value n = 100 into the expression for t_n.

Example

Find the first four terms of the sequence: $d_n = 4.9n^2, n \in \{1,2,3,\dots\}$

Solution

$d_1 = 4.9 \times 1^2 = 4.9$, $d_2 = 4.9 \times 2^2 = 19.6$, $d_3 = 4.9 \times 3^2 = 44.1$ and $d_4 = 4.9 \times 4^2 = 78.4$.
The sequence is $\{4.9, 19.6, 44.1, 78.4, \dots\}$

Exercise Set 2

1. Find the first, fifth and tenth terms in the following sequences
   a. $t_n = 2n - 5, n \in \{1,2,3,\dots\}$
   b. $t_n = \frac{n}{n+1}, n \in \{1,2,3,\dots\}$
   c. $t_n = (-1)^n + n, n \in \{1,2,3,\dots\}$
   d. $t_n = n^2 - n + 41, n \in \{1,2,3,\dots\}$

3. Iterative definition

An iterative definition is expressed in the form: $t_{n+1} = 3t_n - 2, t_1 = 6$. This definition looks complicated, but it is actually straight forward. The word iteration means the calculation of the next term from the previous ter, using the same procedure. The symbol t_{n+1} simply means the next term after the term t_n.

Example

Find the first four terms of the sequence above: $t_{n+1} = 3t_n - 2, t_1 = 6$

Solution

The first term, t_1, is 6 (this is given in the definition).
The second term, t_2, is $3 \times 6 - 2 = 16$
The third term, t_3, is $3 \times 16 - 2 = 46$
The fourth term, t_3, is $3 \times 46 - 2 = 136$

Exercise Set 3

1. Find the first 4 terms in the following sequences.
   a. $t_{n+1} = t_n + 2, t_1 = 3$
   b. $t_{n+1} = 3t_n, t_1 = 2$
   c. $t_{n+1} = t_n - 7, t_1 = 14$
   d. $t_{n+2} = t_{n+1} + t_n, t_1 = 1, t_2 = 1$

Arithmetic Sequences

An arithmetic sequence is a sequence where there is a common different between any two successive terms.

For example: 2, 5, 8, 11, 14, … is arithmetic as $5 - 2 = 8 - 5 = 11 - 8 = 14 - 11$, etc, because they each have a difference of 3.
Likewise, 31, 27, 23, 19, … is arithmetic as $31 - 27 = 27 - 23 = 23 - 19$, etc, because they each have a difference of -4.

Algebraically, $\boldsymbol{T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = \cdots = T_{n+1} - T_n}$

Why 'arithmetic'

If a, b and c are any three consecutive terms of an arithmetic sequence then $b - a = c - b$ (equating common differences). Therefore, through rearranging $2b = a + c$, which gives $b = \frac{a+c}{2}$. The middle term is the arithmetic mean (average) of the terms on each side of it. Hence the name arithmetic sequence.

The General Term Formula

In an arithmetic sequence, the first term is denoted by a and the common difference by d. The position of the term is denoted by n, such as for the first term $n - 1$, second term $n - 2$, third term $n - 3$, etc.

Consider the sequence 2, 9, 16, 23, 30, … For this sequence, $a = 2$ and $d = 7$

2	9	16	23
a	$a + d$	$a + d + d$	$a + d + d + d$
a	$a + d$	$a + 2d$	$a + 3d$

Thus, for a given position, n, the term is given by $a + (n - 1)d$

Therefore, the general term of an arithmetic progression is given by:

$$\boldsymbol{T_n = a + (n - 1)d}$$

For the sequence above, the formula for its general term is given by $T_n = 2 + (n - 1) \times 7$
So, the 50th term is $2 + (50 - 1) \times 7 = 2 + 49 \times 7 = 2 + 343 = 345$

Exercise Set 4

1. Consider the sequence 6, 17, 28, 39, 50, …
   a. Show that the sequence is arithmetic (Check for a common difference).

   b. Find the formula for its general term.

   c. Find the 50th term.

   d. Is 325 a member of the sequence?

2. Consider the sequence 87, 83, 79, 75, …
   a. Show that the sequence is arithmetic.

b. Find the formula for the general term.

c. Find the 40^{th} term.

d. Is -143 a member of the sequence?

3. Show that the following sequences are arithmetic

a. {-0.12, 3.48, 7.08, …}

b. $\left\{\frac{5}{9}, -\frac{1}{9}, -\frac{7}{9}, \ldots\right\}$

c. $\left\{5\frac{2}{3}, 7\frac{4}{15}, 8\frac{13}{15}, \ldots\right\}$

d. $\{x+9, 2x+7, 3x+5, \ldots\}$

4. For the arithmetic sequence {22, m, n, 37, …}, find the values for m and n.

Example

Find k given that 3k+1, k and -3 are consecutive terms of an arithmetic sequence.

Solution

Since the terms are consecutive, $\mathrm{k}-(3\mathrm{k}+1)=-3-\mathrm{k}$ (2^{nd} term – 1^{st} term = 3^{rd} term – 2^{nd} term).
$\mathrm{k}-3\mathrm{k}-1=-3-\mathrm{k}$ (removing of brackets)
$-2\mathrm{k}-1=-3-\mathrm{k}$ (collecting like terms)
$-1+3=2\mathrm{k}-\mathrm{k}$ (rearranging)
Therefore, $\mathrm{k}=2$

5. Find k given the consecutive arithmetic terms:

a. 32, k, 3

b. $\mathrm{k}+1, 2\mathrm{k}+1, 13$

Example

Find the general term T_n for an arithmetic sequence with $T_3 = 8$ and $T_8 = -17$

Solution

$T_3 = 8$ which tells us, $a + 2d = 8 \ldots$ (1) using $T_n = a + (n-1)d$
$T_8 = -17$ which tells us, $a + 7d = -17 \ldots$ (2)
We now solve (1) and (2) simultaneously

$a + 2d = 8$
$a + 7d = -17$
(1)-(2) gives $-5d = 25$
Therefore $d = -5$

Substitute $d = -5$ into (1) gives
$a + 2(-5) = 8$
$a - 10 = 8$
$a = 18$

Therefore, the sequence is 18, 13, 8, 3, -2, etc and the general equations is $T_n = 18 + (n-1) \times -5$

6. Find the general term T_n for an arithmetic sequence given that:
   a. $T_7 = 41$ and $T_{13} = 77$

   b. The seventh term is 1 and fifteenth term is -39