WEEK 6 NOTES & EXERCISES

The sum of a given number of terms of an arithmetic sequence

When the terms of an arithmetic sequence are added together, an arithmetic series is formed.
So, 5, 9, 13, 17, 21, . . . is an arithmetic sequence
whereas 5 + 9 + 13 + 17 + 21 + . . . is an arithmetic series.
The sum of n terms of an arithmetic sequence is given by S_n.
The formula for the sum of n terms of an arithmetic sequence when the value of a and d are known is given by;

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

Example

Find the sum of 4 + 7 + 10 + 13 + ….. to 50 terms.

The series is arithmetic with a = 4, d = 3 and n = 50

Using $S_n = \frac{n}{2}\{2a + (n+1)d\}$ gives

$$S_n = \frac{50}{2}\{2 \times 4 + (50-1) \times 3\}$$
$$= 25\{8 + 49 \times 3\}$$
$$= 3875$$

Exercise Set 1

Q1. a) Find the sum of 3 + 7 + 11 + 15 + …..to 20 terms

b) Find the sum of 100+93+86+79+…. to 40 terms

c) Find the sum of $\frac{1}{2} + 3 + 5\frac{1}{2} + 8 + \ldots$ to 50 terms.

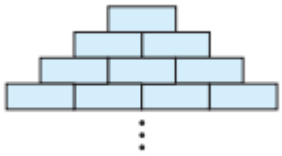
Q2. Find the sum of the first 100 positive integers.

Q3. A sequence is 5, 7, 9, 11, . . . How many consecutive terms need to be added to obtain 357?

Q4. The first term in an arithmetic sequence is 5 and the sum of the first 20 terms is 1240. Find the common difference, d.

Q5. The first term is 50 and the 10th term is −40. Find S10.

Q6. Find the sum of 5 + 8 + 11 + 14 + … + 101. Note: First use $T_n = a + (n - 1)d$ to find the value of n.

Q7. 

A bricklayer builds a triangular wall with layers of bricks as shown. If the bricklayer uses 171 bricks, how many layers are placed?

Geometric sequences

A farmer is breeding worms which he hopes to sell to local shire councils for use in the decomposition of waste at rubbish dumps. Worms reproduce readily and the farmer expects a 10% increase per week in the mass of worms that he is farming. A 10% increase per week would mean that the mass of worms would increase by a constant factor of $(1 + \frac{1}{10})$ or 1.1. He starts off with 10 kg of worms. By the beginning of the second week he will expect 10 × 1.1 = 11 kg of worms, by the start of the third week he would expect 11 × 1.1 = 10 × $(1.1)^2$ = 12.1 kg of worms, and so on. This is an example of a geometric sequence.

A geometric sequence is one whereby the first term is multiplied by a number, known as the common ratio, to create the second term which is multiplied by the common ratio to create the third term, and so on. The first term in a geometric sequence is referred to as *a* and the common ratio is referred to as *r*. Consider the geometric sequence where *a* = 1 and *r* = 3. The terms in the sequence are:

$$1 \xrightarrow{\times 3} 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27 \xrightarrow{\times 3} 81..$$

To discover the common ratio, r, of a geometric sequence you need to calculate the ratio of successive terms, namely $\frac{t_2}{t_1}$. You could alternatively calculate $\frac{t_3}{t_2}$ or $\frac{t_4}{t_3}$ and so on.

A geometric sequence is a sequence of numbers for which the ratio of successive terms is the same.

$$\frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3} = \ldots = \textit{common ratio}$$

The first term of a geometric sequence is referred to as *a*.
The common ratio between a term and its preceding term is referred to as *r*.

Exercise Set 2

Q1. Which of the following are geometric sequences?

a) 1, 2, 4, 8, 16, . . .

b) 2, 6, 18, 54, 162, . . .

c) 1, 4, 16, 64, 256, . . .

d) 2, 6, 12, 24, 48, . . .

e) 1, 5, 25, 100, 125, . . .

f) 0, 2, 4, 8, 16, . . .

g) −2, 4, −8, 16, −32, . . .

h) −1, −5, 10, −20, −40, . . .

i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

Q2. Look again at the geometric sequences found in Q1. Write down the values of a and r.

a) b) c)

d) e) f)

g) h) i)

Q3. On the first day Jenny hears a rumour. On the second day, she tells two friends. On the third day, each of these two friends tell two of their own friends, and so on.

a) Write the geometric sequence for the first five days of the above real-life situation.

b) Find the value of r.

c) How many people are told of the rumour on the 12th day?

d) How many people altogether have heard the rumour on the 12th day?

Finding the terms of a geometric sequence

Consider the finite geometric sequence of seven terms for which $a = 3$ and $r = 4$.

$$3 \xrightarrow{\times 4} 12 \xrightarrow{\times 4} 48 \xrightarrow{\times 4} 192 \xrightarrow{\times 4} 384$$

Now,

$t_1 = 3$	$t_1 = a$	
$t_2 = 3 \times 4$	$t_2 = a \times r$	$t_2 = a \times r^1$
$t_3 = 3 \times 4 \times 4$	$t_3 = a \times r \times r$	$t_3 = a \times r^2$
$t_4 = 3 \times 4 \times 4 \times 4$	$t_4 = a \times r \times r \times r$	$t_4 = a \times r^3$
$t_5 = 3 \times 4 \times 4 \times 4 \times 4$	$t_5 = a \times r \times r \times r \times r$	$t_5 = a \times r^4$ and so on . . .

We notice a pattern emerging. That pattern can be described by the equation:

$$t_n = 3 \times 4^{n-1}.$$

For example, if $n = 5$,

$$t_5 = 3 \times 4^4.$$

We can generalise this rule for all geometric sequences.

$$t_n = ar^{n-1}$$

where t_n is the nth term
a is the first term
r is the common ratio.

This rule enables us to find any term of a geometric sequence provided we know the value of a and r.

Example

Find the 12th term of the geometric sequence: 2, 10, 50, 250, 1250, . . .

1. Find the value of a. $a = 2$
2. It has been stated that it is a geometric sequence, so find the value of r. $r = \frac{10}{2} = 5$
3. Use the rule $t_n = a \times r^{n-1}$ to find the 12th term. $t_{12} = 2 \times 5^{12-1} = 97\,656\,250$
4. Write your answer. The value of the 12th term is 97 656 250.

Exercise Set 3

Q1. Find the value of the term specified for the given geometric sequences.

a) Find the 15th term of the geometric sequence 2, 8, 32, 128, 512, . . .

b) Find the 10th term of the geometric sequence 2, 12, 72, 432, 2592, . . .

c) Find the 20th term of the geometric sequence 1.1, 2.2, 4.4, 8.8, 17.6, . . .

d) Find the 8th term of the geometric sequence 3.1, 8.06, 20.956, 54.4856, 141.662 56, . . .

e) Find the 9th term of the geometric sequence −2, −8, −32, −128, −512, . . .

f) Find the 10th term of the geometric sequence $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

Example

The 2nd term of a geometric sequence is 8 and the 5th is 512. Find the 10th term of this sequence.

1 We know that $t_2 = 8$ and that $t_n = a \times r^{n-1}$.	$t_2 = a \times r^1$ $= 8$	
2 We know that $t_5 = 512$ and that $t_n = a \times r^{n-1}$.	$t_5 = a \times r^4$ $= 512$	
3 Solve the 2 equations simultaneously by eliminating a, to find r.	$a \times r^1 = 8$ $a \times r^4 = 512$	[1] [2]
Divide equation 2 by equation 1.	$\frac{a \times r^4}{a \times r} = \frac{512}{8}$ $r^3 = 64$ $r = 4$	[2] ÷ [1]
4 To find a, substitute the value of r.	Substitute $r = 4$ into equation [1]. $a \times 4 = 8$ $a = 2$	
5 Write down the rule.	$t_n = 2 \times 4^{n-1}$	
6 To find the 10th term, let $n = 10$.	$t_{10} = 2 \times 4^9$ $= 524\,288$	
7 Write your answer.	The 10th term in the sequence is 524 288.	

Q2. Find the value of the term specified for the specified geometric sequences.

a) The 2nd term of a geometric sequence is 6 and the 5th term is 162. Find the 10th term.

b) The 2nd term of a geometric sequence is 6 and the 5th term is 48. Find the 12th term

Q3. For the geometric sequence 3, m, n, 192, . . . , find the values for m and n

Q4. The first three terms of a geometric sequence are 2, 6, and 18. Which numbered term would be the first to exceed 1 000 000 in this sequence? You will need to use trial and error to solve this.

Q5. Redo Q3 c) from Exercise Set 1 using the formula for the term of a GP (geometric progression).