MATHEMATICAL APPLICATIONS 3

WEEK 5 NOTES & EXERCISES

Sequences and series

NUMBER PATTERNS

An important skill in mathematics is to be able to

- recognise patterns in sets of numbers,
- describe the patterns in words, and
- continue the patterns.

Sequences of numbers play an important part in our everyday life. For example, the following sequence: 2.25, 2.37, 2.58, 2.57, 2.63, . . . gives the end-of-day trading price (for 5 consecutive days) of a share in an electronics company. It looks like the price is on the rise, but is it possible to accurately predict the future price per share of the company?

The following sequence is more predictable:

10 000, 9000, 8100, . . .

This is the estimated number of radioactive decays of a medical compound each minute after administration to a patient. The compound is used to diagnose tumours. In the first minute, 10 000 radioactive decays are predicted; during the second minute, 9000, and so on. Can you predict the next number in the sequence? You're correct if you said 7290. Each successive term here is 90% of, or 0.90 times, the previous term.

Sequences are strings of numbers. They may be finite in number or infinite. Number sequences may follow an easily recognisable pattern or they may not. A great deal of recent mathematical work has gone into deciding whether certain strings follow a pattern (in which case subsequent terms could be predicted) or whether they are random (in which case subsequent terms cannot be predicted). This work forms the basis of chaos theory, speech recognition software for computers, weather prediction and stock market forecasting, to name but a few uses.

Sequences which follow a pattern can be described a number of different ways. They may be listed in sequential order, they may be described as a functional definition, or they may be described in an iterative definition.

1 Listing in sequential order

3, 7, 11, 15, ... forms a number sequence. The first term is 3, the second term is 7, the third term is 11, etc.

We describe this pattern in words:

"The sequence starts at 3 and each term is 4 more than the previous one."

Thus, the fifth term is 19, and the sixth term is 23, etc.

Example

Describe the sequence: 14, 17, 20, 23, ... and write down the next two terms.

The sequence starts at 14 and each term is 3 more than the previous term. The next two terms are 26 and 29.

Exercise Set 1

Q1. Write down the first four terms of the sequence if you start with:

a) 4 and add 9 each time

b) 45 and subtract 6 each time

c) 2 and multiply by 3 each time

d) 96 and divide by 2 each time.

Q2. Describe the following number patterns and write down the next 3 terms:

a) 1, 4, 9, 16, ...

b) 1, 8, 27, 64, ...

2 Functional definition

A functional definition is expressed in the form: $t_n = 2n - 7,\ n \in \{1, 2, 3, 4, \ldots\}$ Using this definition the nth term can be readily calculated. For this example $t_1 = 2 \times 1 - 7 = -5$, $t_2 = 2 \times 2 - 7 = -3$, $t_3 = 2 \times 3 - 7 = -1$ and so on. We can readily calculate the 100th term, $t_{100} = 2 \times 100 - 7 = 193$, simply by substituting the value n = 100 into the expression for t_n.

Look at the following example: $d_n = 4.9n^2$, $n \in \{1, 2, 3, \ldots\}$ For this example, in which the sequence is given the name d, $d_1 = 4.9 \times 1^2 = 4.9$, $d_2 = 4.9 \times 2^2 = 19.6$. Listing the sequence would yield d: {4.9, 19.6, 44.1, 78.4, . . .}. The 10th term would be $4.9 \times 10^2 = 490$.

Exercise Set 2

Q1. Find the first, fifth and tenth terms in the following sequences.

a) $T_n = 2n - 5,\ n \in \{1, 2, 3, \ldots\}$

b) $t_n = \frac{n}{n+1},\ n \in \{1, 2, 3, \ldots\}$

c) $t_n = tn = 17 - 3.7n,\ n \in \{1, 2, 3, \ldots\}$

d) $t_n = 5 \times \left(\frac{1}{2}\right)^n,\ n \in \{1, 2, 3, \ldots\}$

e) $t_n = (-1)^n + n, n \in \{1, 2, 3, . . .\}$

f) $t_n = n^2 - n + 41, n \in \{1, 2, 3, . . .\}$

3 Iterative definition

An iterative definition is expressed in the form: $t_n + 1 = 3t_n - 2; t_1 = 6$. This definition looks complicated, but is actually straightforward. The word iteration means the calculation of the next term from the previous term using the same procedure. The symbol t_{n+1} simply means the next term after the term t_n. In the above example the first term, t_1, is 6 (this is given in the definition) and so the next term, t_2, is $3 \times 6 - 2 = 16$, and the following term is $3 \times 16 - 2 = 46$. In each and all cases the next term is found by multiplying the previous term by 3 and then subtracting 2. We could write the sequence out as a table:

t_n	Comment
$t_1 = 6$	Given in the definition
$t_2 = 3t_1 - 2$ $= 3 \times 6 - 2$ $= 16$	Using t_1 to find the next term, t_2
$t_3 = 3t_2 - 2$ $= 3 \times 16 - 2$ $= 46$	Using t_2 to find the next term, t_3
$t_4 = 3t_3 - 2$ $= 3 \times 46 - 2$ $= 136$	Using t_3 to find the next term, t_4

Exercise Set 3

Find the first 4 terms in the following sequences.

a) $u_{n+1} = u_n + 2, u_1 = 3$

b) $u_{n+1} = 3u_n, u_2 = 2$

c) $u_n + 1 = un - 7, u1 = 14$

d) $u_{n+2} = u_{n+1} + u_n, u_1 = 1, u_2 = 1$

Arithmetic Sequences

An arithmetic sequence is a sequence where there is a common difference between any two successive terms.

For example: 2, 5, 8, 11, 14, ... is arithmetic as 5 – 2 = 8 – 5 = 11 - 8 = 14 - 11, etc. Difference = 3
Likewise, 31, 27, 23, 19, ... is arithmetic as 27 - 31 = 23 - 27 = 19 - 23, etc. Difference = -4

Algebraically, $T_2 - T_1 = T_3 - T_2 = T_4 - T_3 \ldots = T_{n++1} - T_n$

Why 'Arithmetic'

If a, b and c are any consecutive terms of an arithmetic sequence then b - a = c - b (equating common differences). Thus 2b = a + c which gives $b = \frac{a+c}{2}$ i.e., middle term = arithmetic mean (average) of terms on each side of it. Hence the name arithmetic sequence.

The General Term Formula

In an arithmetic sequence the first term is denoted by *a* and the common difference by *d*. The position of the term is denoted by *n*. ie for the first term n = 1, for the second term n = 2, for the third n = 3 etc

Consider the sequence 2, 9, 16, 23, 30, ... For this sequence a = 2 and d = 7

2	9	16	23 ……
a	a + d	a + d + d	a + d + d + d
a	a + d	a + 2d	a + 3d

Thus, for a given position, n, the term is given by a + (n-1)d

The general term of an arithmetic progression is given by

$$T_n = a + (n - 1)d$$

For the sequence above, the formula for its general term is given by $T_n = 2 + (n – 1)\times 7$
Thus the 50^{th} term is $2 + (50 – 1)\times 7 = 2 + 49\times 7 = 2 + 343 = 350$

Exercise Set 4

Q1. Consider the sequence 6, 17, 28, 39, 50, ...

a) Show that the sequence is arithmetic. (Check for a common difference)

b) Find the formula for its general term.

c) Find its 50th term.

d) d Is 325 a member?

e) e Is 761 a member?

Q2. Consider the sequence 87, 83, 79, 75, ...

a) Show that the sequence is arithmetic.

b) Find the formula for the general term.

c) Find the 40th term.

d) Is -143 a member?

Q3. Show that the following sequences are arithmetic.

a) $\{-0.12, 3.48, 7.08, \ldots\}$

b) $\{\frac{5}{9}, -\frac{1}{9}, -\frac{7}{9}, \ldots\}$

c) $\{5\frac{2}{3}, 7\frac{4}{15}, 8\frac{13}{15}, \ldots\}$

d) $\{x + 9, 2x + 7, 3x + 5, \ldots\}$

Q4. For the arithmetic sequence {22, m, n, 37, . . .}, find the values for m and n.

Example

Find k given that 3k + 1, k and -3 are consecutive terms of an arithmetic sequence.

Since the terms are consecutive, k - (3k + 1) = -3 - k (ie 2^{nd} term – 1^{st} term = 3^{rd} term – 2^{nd} term)
k - 3k - 1 = -3 - k (removing the brackets)
-2k - 1 = -3 - k
-1+3= -k + 2k
Thus k = 2

Q5. Find k given the consecutive arithmetic terms:

a) 32, k, 3

b) k + 1, 2k + 1, 13

Example

Find the general term T_n for an arithmetic sequence with $T_3 = 8$ and $T_8 = -17$

$T_3 = 8$ which gives $a + 2d = 8$ …..(1) using $T_n = a + (n – 1)d$

$T_8 = -17$ which gives $a + 7d = - 17$ … (2)

We now solve (1) and (2) simultaneously

$a + 2d = 8$

$a + 7d = -17$

(1) – (2) gives

$-5d = 25$

Thus $d = -5$

Substituting in (1) gives

$a + 2(-5) = 8$

$a – 10 = 8$

$a = 18$

thus the sequence is 18, 13, 8, 3, -2 etc and the general equation is

$T_n = 18 + (n – 1)\times-5$

Q6. Find the general term un for an arithmetic sequence given that:

a) $T_7 = 41$ and $T_{13} = 77$

b) the seventh term is 1 and the fifteenth term is -39