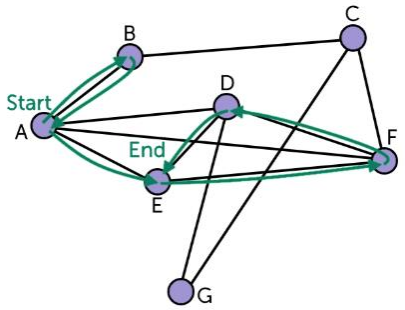
MATHEMATICAL APPLICATIONS 3

WEEK 13 NOTES & EXERCISES

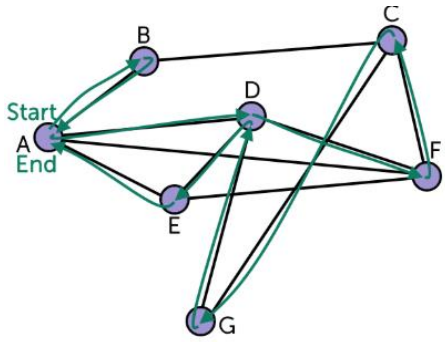
Moving around a network – Walks (Paths)

If we start at a vertex and then move along an edge (in the direction of the arrow, if it's a directed edge) to a new vertex, we are starting a **walk**. We can continue the walk by selecting another edge coming out of the second vertex, then a third edge coming out of the third vertex, and so on. In a simple network we can represent a walk by a sequence of vertices. Here is a simple, undirected network, with arrows representing a walk around the network:



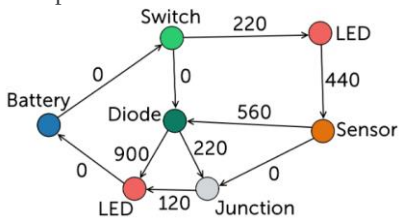
We can record this walk as the sequence ABAEFDE.

A walk that **starts and ends at the same vertex** is called **closed**, and if it **starts and ends at different vertices** it is called **open**. The walk above is open - below is a different walk on the same network:

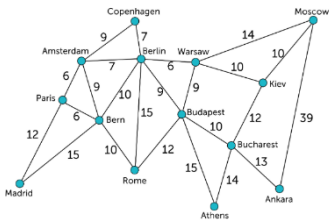


This walk has sequence *ABADFCGDEA*. Since it starts and ends at the same vertex, it is **closed**. We can also tell from just looking at the sequence (and not the diagram) that's the walk is closed, since the sequence **starts and ends with the same letter**.

We represent relationships in a network with edges, and often we will want to **attach a measurement** to each edge to show how the relationships are different. For example, if we were making a network based on the roads between towns, we might want to show more than just which towns are connected - **how long** is each road, or **how long does it take to travel** along each one? To show this kind of extra detail, we **add a number** to each edge, and call this number a **weight**. Here are a few examples.

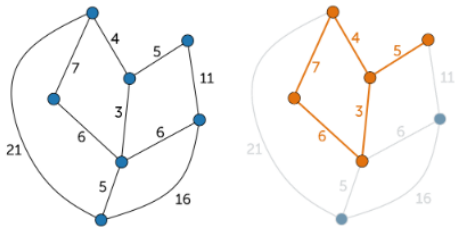


A simple circuit with resistance measured in OHMS.



Approximate travel time, in hours, between some European capital cities.

If a network has weights on its edges, we can give a weight to **the whole network** by adding up all the weights. This applies to subnetworks too. Similarly, we can give a weight to any walk by adding up all the weights on the edges chosen in the walk (the weight of a walk is sometimes called its **length**).

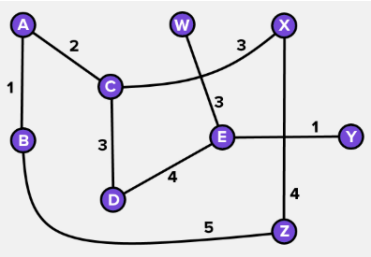


This **network** has a weight of 4+5+7+3+6+6+11+5+16+21=84.

The **subnetwork** in the middle (highlighted) has weight 4+5+7+3+6=25.

Exercise Set 1

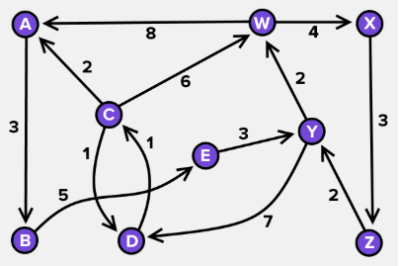
Q1. Find the weights of the following paths.



a) Path A-B-Z-X

b) Path X-C-A-B-Z

Q2. Find the weights of the following paths in this network.



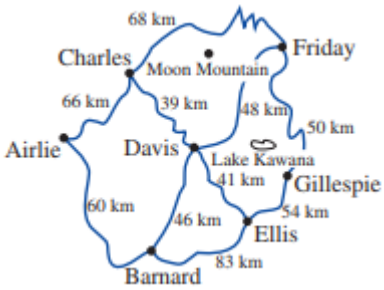
a) The path from *Z* to *C*

b) The path from *B* to *X*

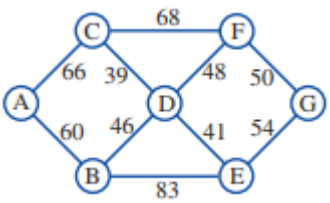
Example

George and Effie want to drive from Airlie to Gillespie using the map at right.

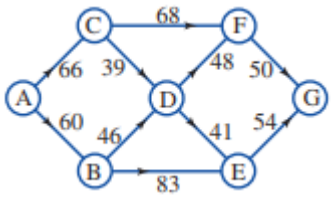
a) Draw a network which represents the map.
b) Given that each road taken must bring them closer to Gillespie, list the number of ways from Airlie to Gillespie. How many ways are possible?
c) Identify the shortest path from the possible routes in b.



Represent towns with circles, called nodes, labelled with the first letter of the town. Ignore the bends in the roads and use straight lines to represent roads connecting the towns.



Each road taken from Airlie must go towards Gillespie. Indicate the direction on each arc with an arrow.



Use the network to list the number of ways from A to G.

A–B–D–E–G
A–B–D–F–G
A–B–E–G
A–C–D–E–G
A–C–D–F–G
A–C–F–G
There are 6 ways to go from Airlie to Gillespie.

Add the lengths of the nodes to calculate the distances of the 6 routes

A–B–D–E–G (60 + 46 + 41 + 54)	201 km
A–B–D–F–G	204 km
A–B–E–G	197 km
A–C–D–E–G	200 km
A–C–D–F–G	203 km
A–C–F–G	184 km

The shortest path is A–C–F–G: Airlie to Charles to Friday to Gillespie.

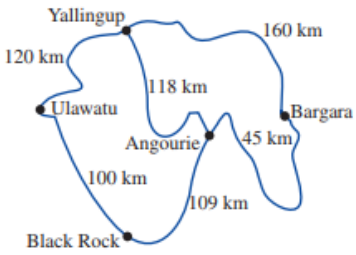
Effie consults the local travel adviser about the travel times for the stages in the journey planned in the example above. She then redraws the network with the average time (in minutes) taken to drive between the towns as shown at right. Which path would take the least time and what is that time?

List all the possible paths and the times they will take.

A-B-D-E-G	153 min
A-B-D-F-G	155 min
A-B-E-G	153 min
A-C-D-E-G	151 min
A-C-D-F-G	153 min
A-C-F-G	156 min

The path ACDEG takes 151 min, the least time.

Q3. A traveller plans a journey from Ulawatu to Bargara (shown on the road map below).



a) Draw a network to represent this situation.

b) Calculate the longest path if no road is travelled twice.

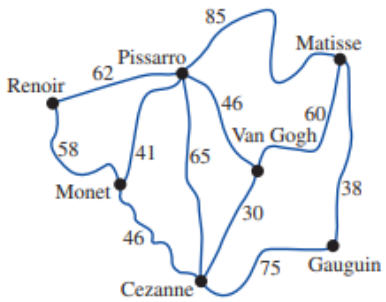
c) The travelling times between each town are: Ulawatu–Yallingup 85 min Ulawatu–Black Rock 75 min Yallingup–Angourie 80 min Black Rock–Angourie 82 min Yallingup–Bargara 120 min Angourie–Bargara 34 min.
Draw a network of this situation showing the time taken to travel between towns on each arc of the network.

d) Calculate the longest time taken to travel from Ulawatu to Bargara.

e) Complete the table showing the shortest distance between each of the towns.

	Ulawatu	Yallingup	Black Rock	Angourie	Bargara
Ulawatu	——	120	100	209	
Yallingup	——	——	220		
Black Rock	——	——	——		
Angourie	——	——	——	——	

Q4. A traveller plans a journey from Renoir to Gauguin. The distances between various nearby towns are shown on the map at right.



a) Calculate the shortest path.

b) The travelling times between the following towns are:

Renoir–Pissarro	47 min
Renoir–Monet	44 min
Monet–Cezanne	40 min
Pissarro–Cezanne	45 min
Pissarro–Van Gogh	34 min
Pissarro–Matisse	75 min
Pissarro–Monet	25 min
Cezanne–Van Gogh	20 min
Van Gogh–Matisse	38 min
Cezanne–Gauguin	59 min
Matisse–Gauguin	28 min

(i) Draw a network of this situation showing the time taken to travel between towns on each arc of the network.

(ii) Calculate the longest time to travel from Renoir to Gauguin.

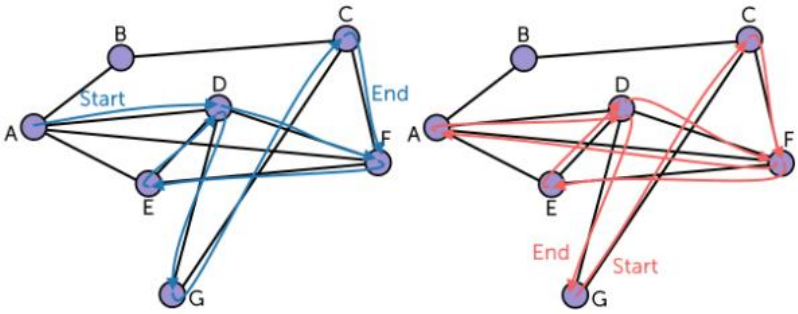
(iii) Calculate the shortest time to travel from Renoir to Gauguin.

c) Complete the table below showing the shortest distance between each of the towns.

	Renoir	Pissarro	Monet	Cezanne	Van Gogh	Matisse	Gauguin
Renoir	—						179
Pissarro	—	—	41				123
Monet	—	—	—				
Cezanne	—	—	—	—			
Van Gogh	—	—	—	—	—		
Matisse	—	—	—	—	—	—	

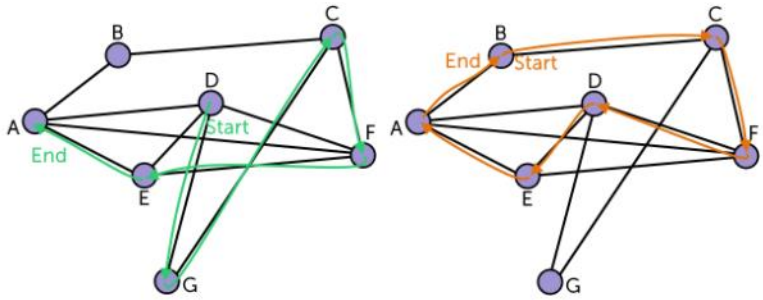
Trails, Paths and Circuits

A walk is the broadest, most general definition of how you can move around a network.
If we say that the walk **cannot use the same edge more than once**, we call it a **trail** (if it's **open**) or a **circuit** (if it's **closed**).
Here is one of each on the same network:



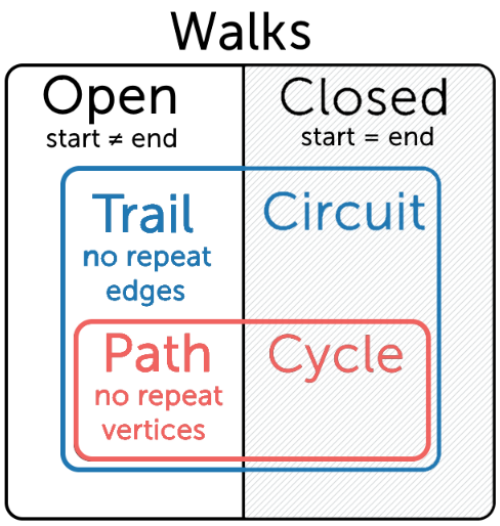
*The walk ADFEDGCF on the left is a **trail**, since no edge is used more than once, and it is open. The walk GCFEDFADG on the right is a **circuit**, since no edge is used more than once, and it is closed.*

If a walk **cannot use the same vertex more than once**, we call it a **path**. If the path is **closed**, we call it a **cycle** (we can reuse the start/end vertex only). Here are one of each on the same network:



*The walk DGCFEA on the left is a **path**, since no vertex is used more than once, and it is open. The Walk BCFDEAB on the right is a **cycle**, since no vertex is used more than once, and it is closed.*

This is summarized as:



*Every sequence of edges is a **walk**. A walk can be **open** (start/end vertices different) or **closed** (start/end vertices same). If edges can't repeat, the walk is a **trail**. A closed trail is a **circuit**. If vertices can't repeat, the walk is a **path**. A closed path is a **cycle**.*

In planar graphs we can define a walk as a sequence of edges and look at various sequences or pathways through the network. Sometimes you may wish to have a walk that use all the edges only once, such as for a road repair gang repairing all the roads in a shire. Sometimes you may wish to have a walk that goes through all the nodes only once, for example, a travelling salesperson who wishes to visit each town only once.

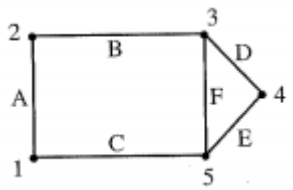
Euler Trails

Consider a walk where every edge is used only once, as in our road repair example.
An Euler trail is one which uses every edge exactly once.

1. For an Euler trail to exist, all vertices must be of an even degree *or* there must be exactly two vertices of odd degree.
2. If the degree of all vertices are even numbers, start with any vertex. In this case the starting vertex and ending vertex are the same.
3. If there are two vertices whose degree is an odd number we use either starting point. The other vertex of odd degree will be the ending point.

Example

Using the figure below, identify an Euler trail.



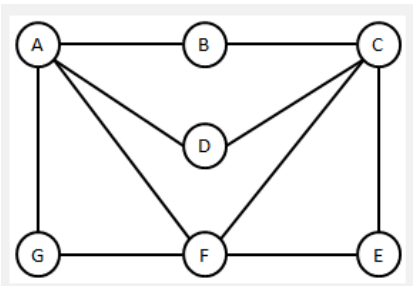
1. Determine a starting vertex. Since there are two odd vertices (3 and 5), use one of these to start eg vertex 3.
2. Attempt to find a trail that uses each edge exactly once. You are allowed to visit a vertex more than once; it is only the edges that are restricted to one use.

   B – A – C – F – D – E or B – A – C – E – D – F or D – E – F – B – A- C are all Euler trails

Note that the starting and ending vertex are not the same, but we started and ended on the vertices with odd degree (3 and 5).

Exercise Set 2

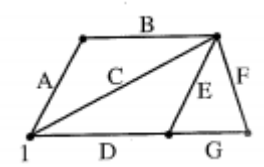
Q1. a) Is it possible to draw an Euler trail on the following network? If so, why?



b) If you were to draw an Euler trail, what vertex would you start with?

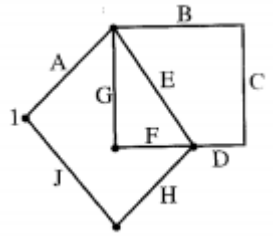
c) Write an Euler circuit starting with B

Q2. a) Using the figure below and starting at vertex 1, identify an Euler trail.



c) Choosing another vertex of degree 3, identify another Euler trail.

Q3. a) Using the figure below and starting at vertex 1, identify an Euler trail.



c) Starting with a vertex of degree 4, identify another Euler trail.

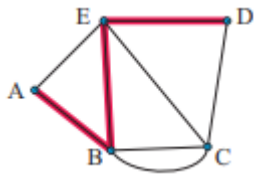
Euler Circuits

With our road repair gang, it would be desirable that the Euler trail started and finished at the same point. This kind of Euler trail is called an Euler circuit: that is, an Euler trail that starts and finishes at the same vertex is an Euler circuit.

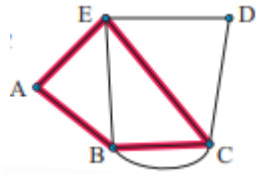
It is important to note that that an Euler circuit cannot exist for planar graphs that have *any* vertices whose degree is odd. In such graphs there is no Euler circuit. Thus the circuit used as an example in 'Euler Trails' does not contain an Euler circuit as vertices 3 and 5 were of odd degree.

Lets gather our thoughts.

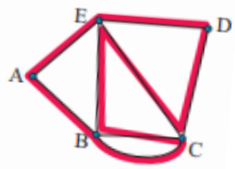
A walk (or path) is a connected sequence of edges or arcs showing a route that starts at one vertex and ends at another. For example: A - B - E - D



A circuit is a walk that starts and ends at the same vertex.
For example: A - B - C - E – A

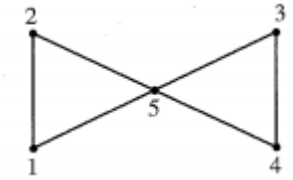


An Eulerian circuit is a circuit that traverses each edge (or arc) exactly once. For example: A-B-C-B-E-C-D-E-A

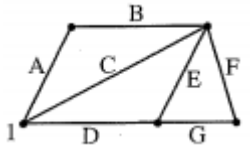


Exercise Set 3

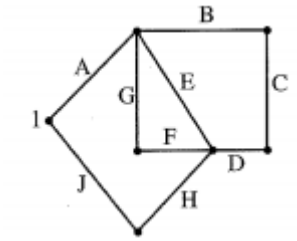
Q1. Using the figure shown and starting at vertex 1, identify an Euler circuit.



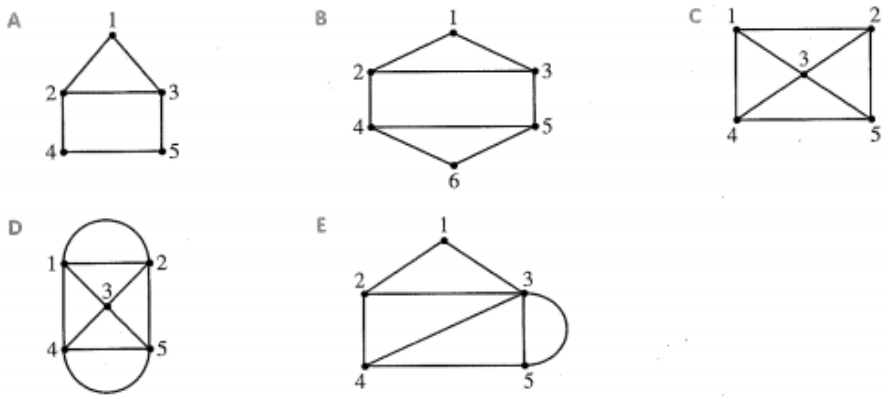
Q2. Is it possible to identify an Euler circuit for the network shown. If not, why not?



Q3. For the network shown, identify an Euler circuit, starting at point 1.



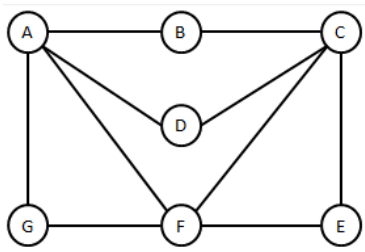
Q4. For which of the following does an Euler circuit exist? Explain.



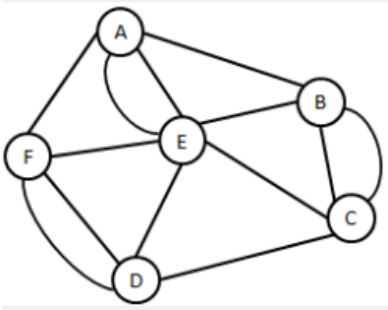
Q5. For the network shown;

a) Why is it possible to draw an Euler circuit?

b) Write an Euler circuit starting at B.



Q5. Write a Euler circuit for the following network.



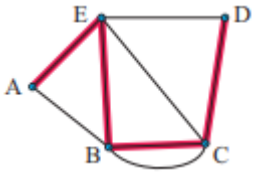
Hamiltonian paths and circuits

In Euler trails and circuits each edge was used exactly once, while vertices could be reused. Now, consider the case where it is desirable to use each vertex *exactly once*, as in the travelling salesperson example.

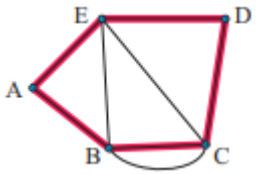
A Hamiltonian path uses every vertex exactly once. It is important to note that not all edges need to be used. Furthermore, there can only be up to 2 vertices with degree 1 (dead ends). In this case these would be the start and/or the finishing vertices.

When determining a Hamiltonian path, sometimes it is desirable to start and finish with the same vertex. For example, our travelling salesperson may live in one of the towns and would like to start and finish in her home town after visiting all the other towns once.
A Hamiltonian circuit is a Hamiltonian path that starts and finishes at the same vertex.

A Hamiltonian path is a path which starts at one vertex and visits each vertex once and only once. It may not finish where it started.
For example: A - E - B - C – D

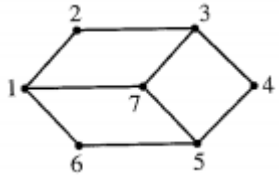


A Hamiltonian circuit is a circuit which starts at one vertex, visits each vertex exactly once, and returns to where it started.
For example: A - B - C - D – E – A

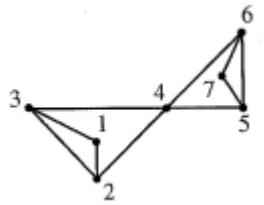


Exercise Set 4

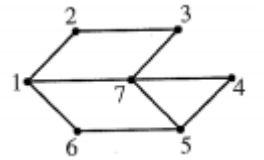
Q1. Starting at vertex 2, determine a Hamiltonian path for the graph shown.



Q2. Starting at vertex 7, determine a Hamiltonian path for the graph shown.



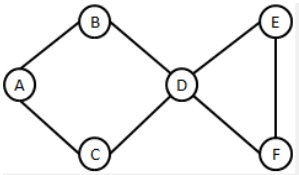
Q3. Starting at vertex 2, determine a Hamiltonian circuit for the graph shown.



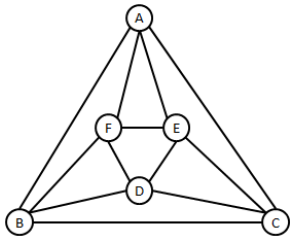
Q4. Which of the following correctly describes a Hamiltonian circuit?

A A circuit that passes through every vertex and every edge exactly once.
B A trail that visits every vertex exactly once.
C A circuit that passes through every edge exactly once.
D A trail that begins and ends at the same vertex and passes through every other vertex exactly once.

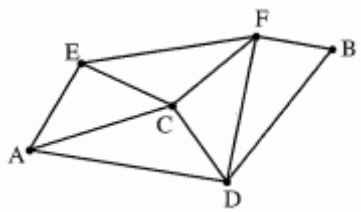
Q5. Is it possible to draw a Hamiltonian circuit of the diagram below? If not, why not?



Q6. Describe a Hamiltonian circuit for the network below, starting at A.



Q7. A PE teacher wishes to plan an orienteering course through a forest following marked tracks. She has placed checkpoints at the points shown in the diagram. The object of any orienteer is to visit each checkpoint once to collect a mark.



a) What path would an efficient orienteer follow if the course starts at C and finishes at B?

b) What path should the orienteer follow if starting and finishing at C?

c) What types of paths are these?