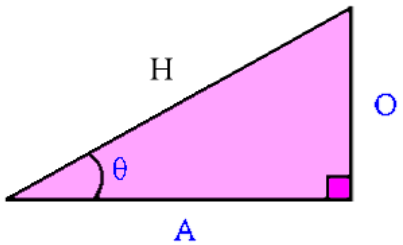
TRIGONOMETRY REVIEW

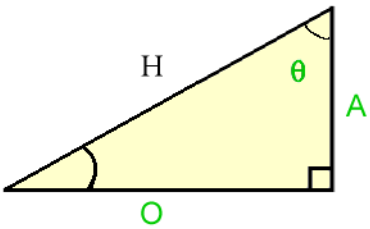
TRIGONOMETRIC RATIOS

If one of the angles of a triangle is 90º (a right angle), the triangle is called a **right angled triangle**. We indicate the 90º (right) angle by placing a box in its corner.) Because the three (internal) angles of a triangle add up to 180º, the other two angles are each less than 90º; that is they are *acute*.



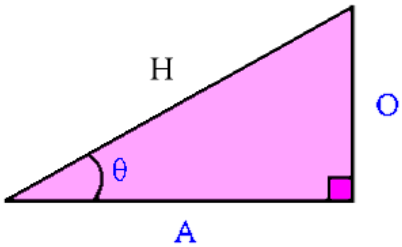
In this triangle, the side H opposite the right angle is called the **hypotenuse**. Relative to the angle θ, the side O, opposite the angle θ is called the *opposite* **side** (it is opposite the angle). The remaining side A is called the *adjacent* **side,** (adjacent means 'next to').

Warning: This assignment of the opposite and adjacent sides is relative to θ. If the angle of interest (in this case θ) is located in the upper right hand corner of the above triangle the assignment of sides is then:

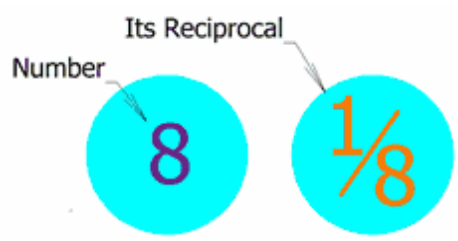


Trigonometric ratios provide relationships between the sides and angles of a right angle triangle. The three most commonly used ratios are:

sine	$sin\theta = \frac{O}{H}$
cosine	$cos\theta = \frac{A}{H}$
tangent	$tan\theta = \frac{O}{A}$ Which is also $= \frac{Hsin\theta}{Hcos\theta} = \frac{sin\theta}{cos\theta}$



RECIPROCAL RATIOS



To get the reciprocal of a number, just divide 1 by the number

Example: the reciprocal of 2 is 1/2 (half)

Every number has a reciprocal except 0 (1/0 is undefined)

It is shown as 1/x, or x^{-1}

If you multiply a number by its reciprocal you get 1

Example: 3 times 1/3 equals 1

Also called the "Multiplicative Inverse"

Other trigonometric ratios are defined by using the original three:

cosecant (cosec)	$csc\theta = \frac{1}{sin\theta} = \frac{O}{A}$
secant (sec)	$sec\theta = \frac{1}{cos\theta} = \frac{H}{A}$
cotangent (cot)	$cot\theta = \frac{1}{tan\theta} = \frac{cos\theta}{sin\theta} = \frac{A}{O}$

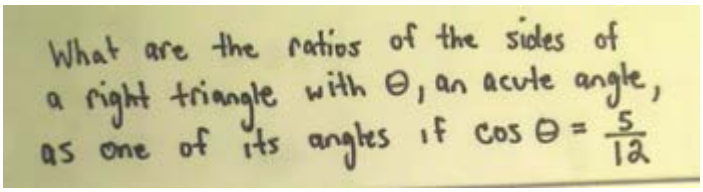
These six ratios define what are known as the trigonometric (trig in short) functions.

Maths Quest 12 Specialist Maths 2A

Q1, Q4, Q6, Q10

FINDING TRIG RATIOS:

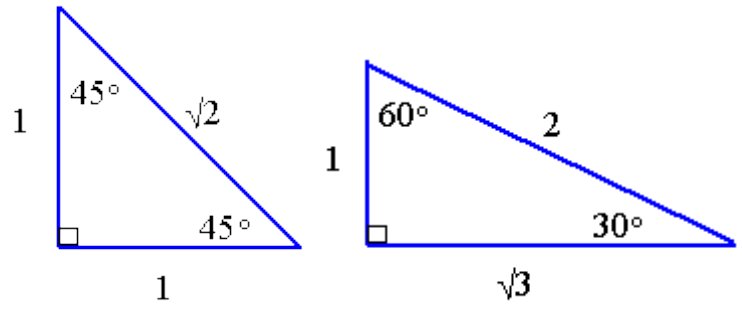
EXAMPLE 1:



watch here to solve it: http://youtu.be/GWdQ9nfyN3Y

EXACT VALUE TRIANGLES

In the topic of trigonometry we have 2 very special triangles called exact value triangles.



These two triangles are very important in the unit, and you will be expected to remember the trigonometric ratios that can be found within them.

They are called exact values, as by using the surds, we have exact values of the relationships created using the angles 45, 30 and 60 degrees.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$ or $\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{\sqrt{3}}{3}$ or $\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

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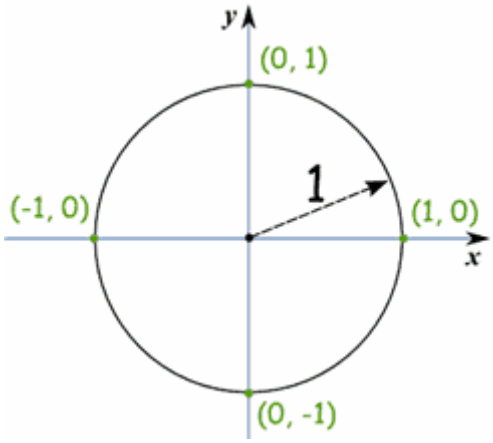
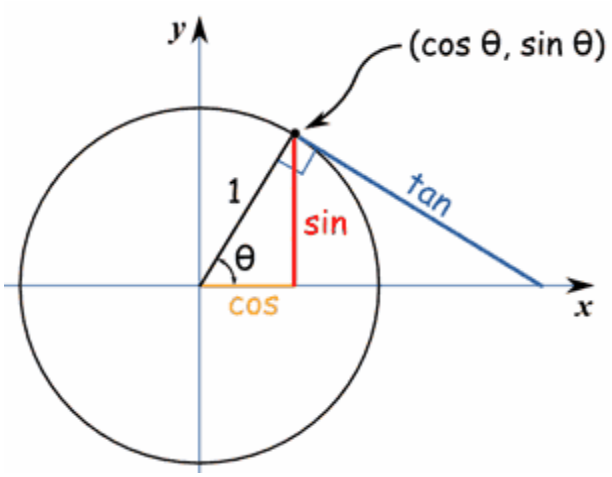
UNIT CIRCLE



The "Unit Circle" is just a circle with a radius of 1.

SINE, COSINE AND TANGENT AND THE UNIT CIRCLE

Because the radius is 1, you can directly measure sine, cosine and tangent.



Cos θ = x (i.e the cos of the angle is equal to the value of the x-coordinate at that point)

This is because, $cos\theta = \frac{A}{H} = \frac{x\ coord}{1} = x\ coordinate$

Sin θ = y (i.e. the sin of the angle is equal to the value of the y-coordinate at that point)

This is because, $sin\theta = \frac{O}{H} = \frac{y\ coord}{1} = y\ coordinate$

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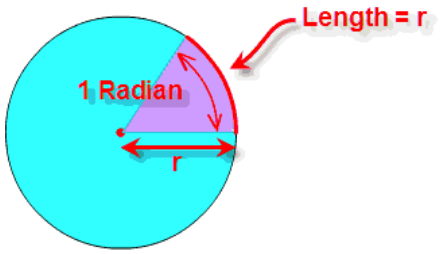
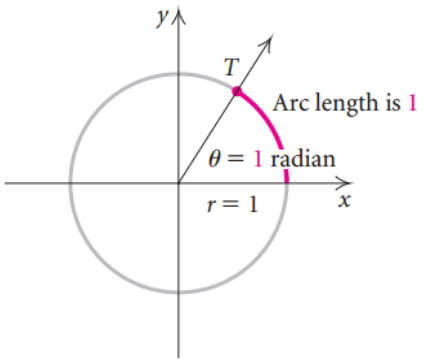
http://www.youtube.com/watch?v=r9UtCf9P7_M&feature=relmfu

http://www.youtube.com/watch?v=IU1KNS0OHwk&feature=relmfu

RADIANS

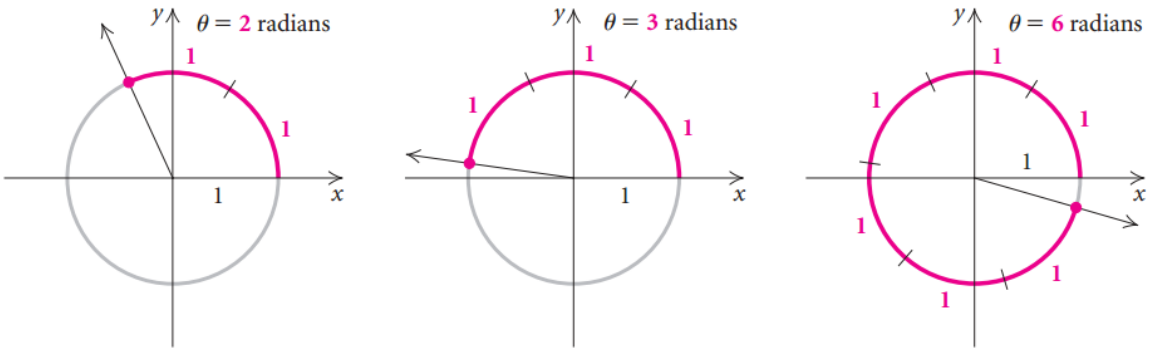




A radian is an angle measure.

It is the angle created when the radius of a circle is wrapped around the circumference.

There are **2π** radians in a full circle, because there are $2\pi r$ in a circumference, which is 2π lots of r, which is 2π lots of radius' which is 2π radians.



If there are **2π** radians in a circle then:

$$2\pi = 360°$$

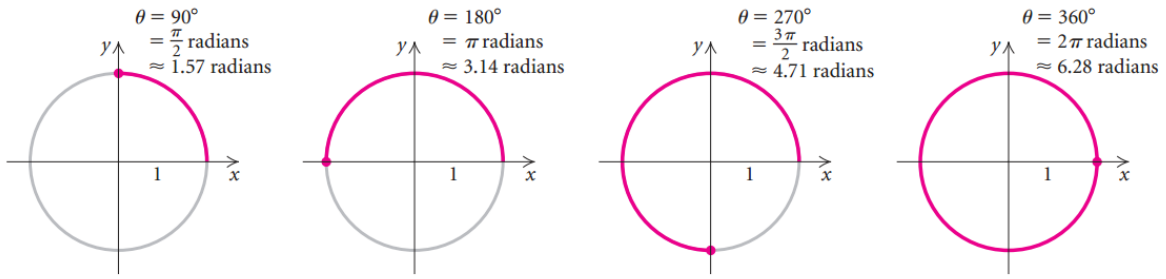
$$\pi = 180°$$

To convert angles to radians:

$$\text{degrees} = \text{radians} \times \frac{180}{\pi}$$

$$\text{radians} = \text{degrees} \times \frac{\pi}{180}$$

Converting between degrees and radians : http://youtu.be/aci0c0dtzGg



You also need to know how to use radians and degrees on your calculator.

Very important is to be fluent in interchanging our exact value angles, 30, 60 and 45 degrees with the radian equivalents.

$30° = \frac{\pi}{6}$

$45° = \frac{\pi}{3}$

$60° = \frac{\pi}{4}$

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Maths Quest 11 Math Methods 6D

Once we know these angles, we also know the exact values for the sin, cos, and tan of these angles using our exact value triangles.

When a rotation, (an angle measured clockwise (if it is positive), or anti clockwise (if it is negative), from the positive x-axis) is given in radians, the word radians is optional and is most often omitted. So if no unit is given for a rotation the rotation is understood to be in radians. This is convention.

http://www.mathcentre.ac.uk/video/?338 this video is quite long (20mins), but is very clear in a description of what radians are. (Methods students need not worry about sector area and arc length, but specialist students should watch those parts carefully)

COMPLEMENTARY ANGLES

Two angles are complementary when the sum of the two angles is $90°$

In a right angle triangle, the two non-right angle measures are complementary.

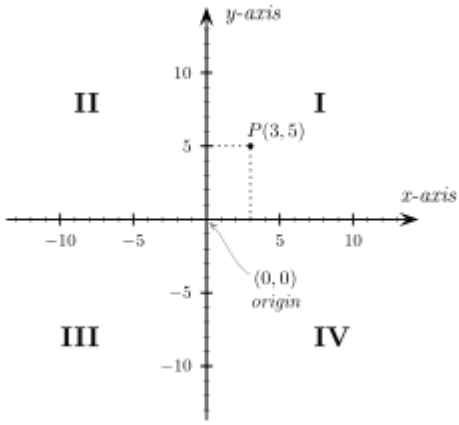
Combining our understanding of right angle triangles, complementary angles and the six trigonometric ratios, we have the following identities.

$$\sin\theta = \cos(90° - \theta), \qquad \cos\theta = \sin(90° - \theta),$$
$$\tan\theta = \cot(90° - \theta), \qquad \cot\theta = \tan(90° - \theta),$$
$$\sec\theta = \csc(90° - \theta), \qquad \csc\theta = \sec(90° - \theta)$$

BOUNDARY VALUES AND QUADRANTS

Typically we break the Cartesian plane up into quadrants using the axis as boundaries.

We label the quadrants 1-4 anti clockwise.

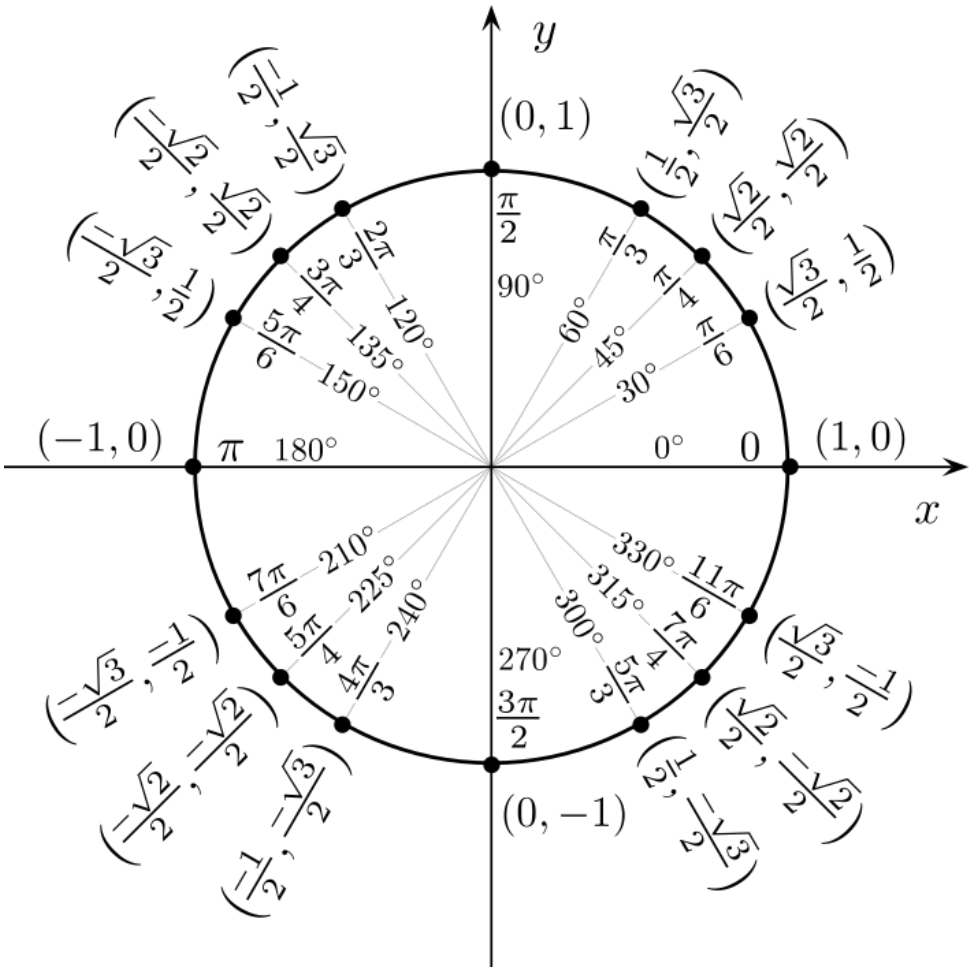


(picture from Wikipedia)

Following our discovery from before where $cos\ \theta = x$ and $sin\ \theta = y$, we can also find values for sin, cos and tan on the boundaries of the quadrants. We need to develop an intuitive sense of $cos = x$, and $sin = y$, and the connection with the unit circle for upcoming work on graphs of trigonometric functions and calculus of trigonometric functions.

http://www.youtube.com/watch?v=DO8DoxwLy8k&feature=relmfu

Coordinate angle	(1,0) 0˚ and 360˚	(0,1) 90˚	(-1,0) 180˚	(0,-1) 270˚
cos (x value)	1	0	-1	0
sin (y value)	0	1	0	-1
tan = (sin/cos)	0	*	0	*
sec (1/cos)	1	*	-1	*
cosec (1/sin)	*	1	*	-1
cot (1/tan= cos/sin)	*	0	*	0

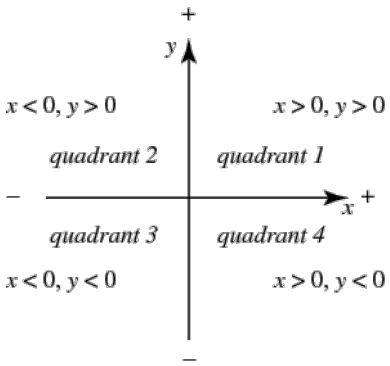


This completed unit circle, shows all the values for our exact value angles, (30, 60, 45) and boundary values. It will be a useful reference tool for you.

Time for a math-interlude: http://youtu.be/YfcIaUF2JqM

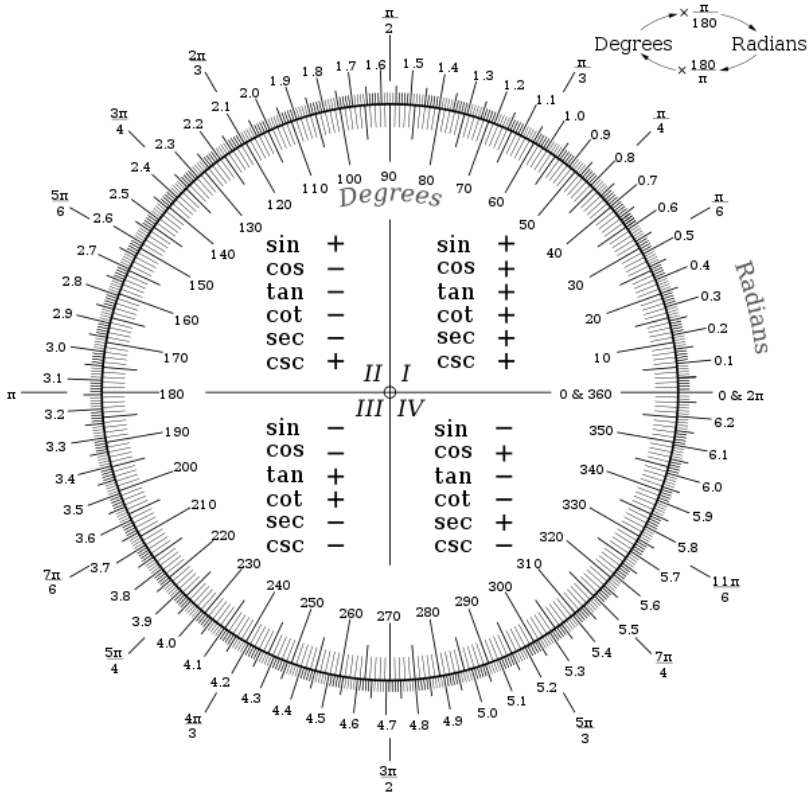
SIGNS IN DIFFERENT QUADRANTS

Remembering that the cosine value in a unit circle is the same as the x-coordinate, we can see that this will mean that in quadrants 2, and 3 that cosine values will be negative. The x-values here are negative.



Similarly we can see that as the sine of a value is related to the y coordinates, that in quadrants 3, and 4 y is negative, and so is the sine values for angles here.

The following diagram summarises the positive and negative status of the 6 trigonometric ratios, it would be useful if you worked through these yourself to confirm.



http://www.youtube.com/watch?v=_gE2aE9OPI8

ARC LENGTH AND AREAS OF SECTORS

If the complete circumference of a circle can be calculated using $\boldsymbol{C = 2\pi r}$ then the length of an arc, (a portion of the circumference) can be found by proportioning the whole circumference.

For example, an arc that spans $\boldsymbol{\pi}$ radians, ($\mathbf{180}°$), is half of the circle, so s (arc length) = $\frac{\boldsymbol{2\pi r}}{\mathbf{2}}$ which is $\boldsymbol{\pi r}$ in length.

To generalise for any angle, consider an arc that spans $\boldsymbol{\theta}$radians. $\boldsymbol{x}$ radians is $\frac{\boldsymbol{\theta}}{\boldsymbol{2\pi}}$ of the whole circle. This means that the arc length will be $\frac{\boldsymbol{\theta}}{\boldsymbol{2\pi}}$ of the whole circumference.

$$\boldsymbol{s = \frac{\theta}{2\pi} \times 2\pi r}$$

$$\boldsymbol{s = \theta r}$$

Similarly for areas of sectors,

The ratio of the area of the sector to the area of the full circle will be the same as the ratio of the angle θ to the angle in a full circle. The full circle has area πr^2. So $\frac{area\ of\ sector}{area\ of\ full\ circle} = \frac{\theta}{2\pi}$, and so the

$$area\ of\ sector = \frac{\theta}{2\pi} \times \pi r^2$$

$$= \frac{1}{2} r^2 \theta$$

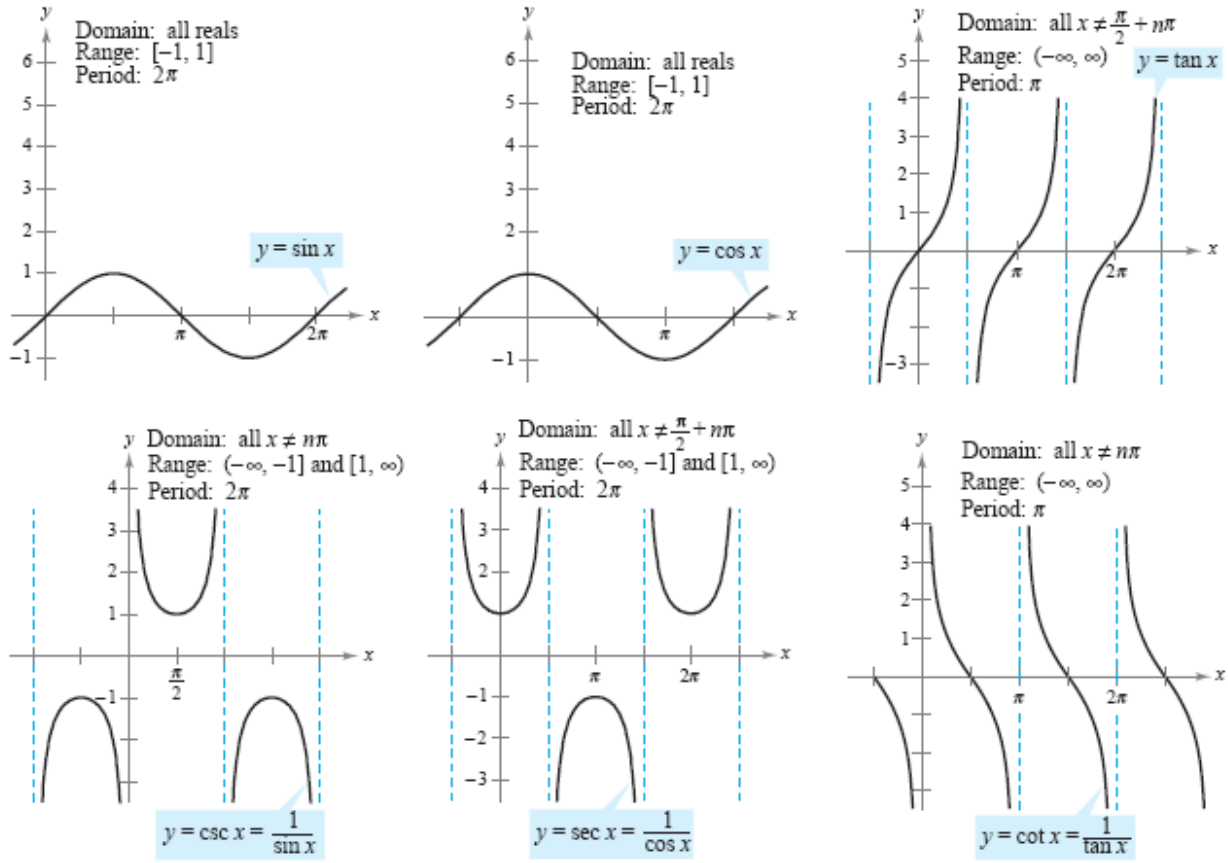
ARC LENGTH $\boldsymbol{s = \theta r}$

AREA OF SECTOR $A_s = \frac{1}{2} r^2 \theta$

TRIGONOMETRIC GRAPHS

First - watch this movie on how trig graphs are constructed out of our knowledge of the unit circle.

http://www.mathcentre.ac.uk/resources/ipod_videos/Trig_ratios_for_any_angle_animation.m4v



The graphs of the six trigonometric functions

As per our exploration with other functions…

The **domain** is: The values that x can take

The **range** is: The values that y can take

Then have a play with this applet on trigonometric graphs.

http://mathinsite.bmth.ac.uk/applet/trig/SinCos.html

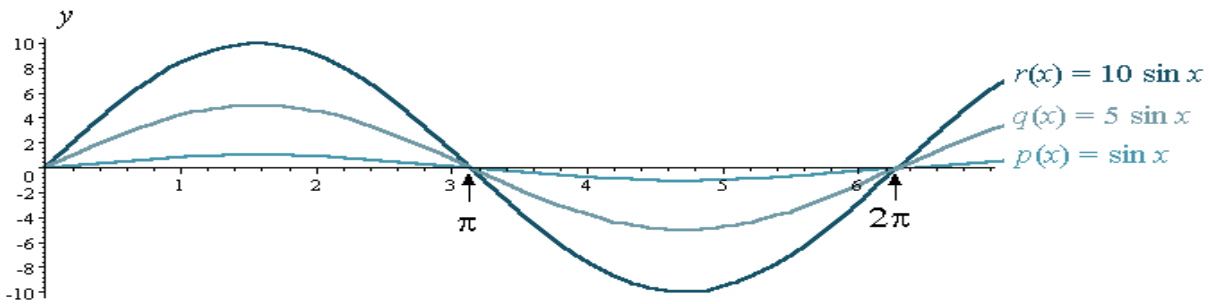
Have this applet open as you work through the following transformations to ensure you understand the movement described.

TRIGONOMETRIC GRAPHS HAVE 4 TYPES OF TRANSFORMATIONS;

AMPLITUDE

The amplitude is the distance from the "resting" position (otherwise known as the mean value or average value) of the curve. Amplitude is always a positive quantity. We could write this using absolute value signs. For the curves y = a sin x, amplitude = |a|.

Here is a Cartesian plane showing the graphs of 3 sine curves with varying amplitudes.



PERIOD

The b in both of the graph types

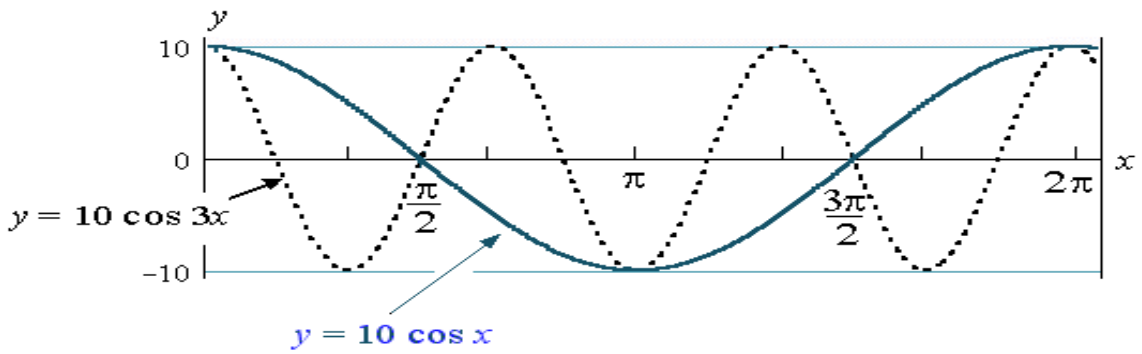
- $y = a \sin bx$
- $y = a \cos bx$

affects the period (or wavelength) of the graph. The period is the distance (or time) that it takes for the sine or cosine curve to begin repeating again.

The period is given by: $\text{Period} = \dfrac{2\pi}{b}$

Note: As b gets larger, the period decreases, b tells us the number of cycles in each 2π.

Here is a Cartesian plane showing the graphs of 2 cosine curves with varying periods, both have amplitude 10.



PHASE SHIFT

Introducing a phase shift, moves us to the following forms of the trig equations:

$$y = a\sin(bx + c)$$

$$y = a\cos(bx + c)$$

Both b and c in these graphs affect the phase shift (or displacement), given by:

$$\text{Phase shift} = \frac{-c}{b}$$

The phase shift is the amount that the curve is moved in a horizontal direction from its normal position. The displacement will be to the left if the phase shift is negative and to the right if the phase shift is positive. This is similar to a horizontal transformation we have seen with other functions.

There is nothing magic about this formula. We are just solving the expression in brackets for zero; bx + c = 0.

NB: Phase angle is not always defined the same as *phase shift*.

VERTICAL TRANSLATION

Vertical translations can still occur with trigonometric functions. This is where we move the whole trig curve up or down on the y-axis. The following two curves have a vertical translation of D units

$$y = a\sin(bx + c) + D$$

$$y = a\cos(bx + c) + D$$

EXAMPLE 2:

Identify the amplitude, period, phase shift and vertical shift for:

1)

$$y = 5 - 3\sin 2(\theta - \frac{\pi}{2})$$

amplitude = |-3| = 3
period = 2π/2 = π
phase shift = π/2 (to the right)
vertical shift = 5

2)

$$y = 2\sin(2x + \frac{\pi}{2})$$

Rewrite

$$y = 2\sin(2x + \tfrac{\pi}{2}) \text{ as } y = 2\sin 2(x + \tfrac{\pi}{4})$$

amplitude = 2

period = π

phase shift = $\frac{\pi}{4}$ units to the left.

vertical shift = none

EXAMPLE 3:

Find the amplitude, period, and phase shift of $y = 3\cos(2x - \pi)$.

Solution: The amplitude is 3, the period is $\frac{2\pi}{2} = \pi$, and the phase shift is $\frac{\pi}{2}$. The graph is shown in Figure 5.2.12:

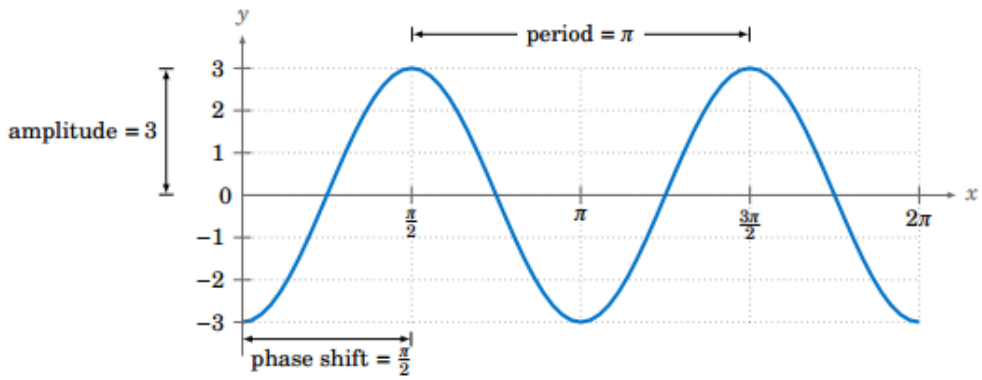


Figure 5.2.12 $y = 3\cos(2x - \pi)$

Notice that the graph is the same as the graph of $y = 3\cos 2x$ shifted to the right by $\frac{\pi}{2}$, the amount of the phase shift.

EXAMPLE 4:

Find the amplitude, period, and phase shift of $y = -2\sin\left(3x + \frac{\pi}{2}\right)$.

Solution: The amplitude is 2, the period is $\frac{2\pi}{3}$, and the phase shift i
sign in the phase shift, since $3x + \pi = 3x - (-\pi)$ is in the form $\omega x - \phi$
5.2.13:

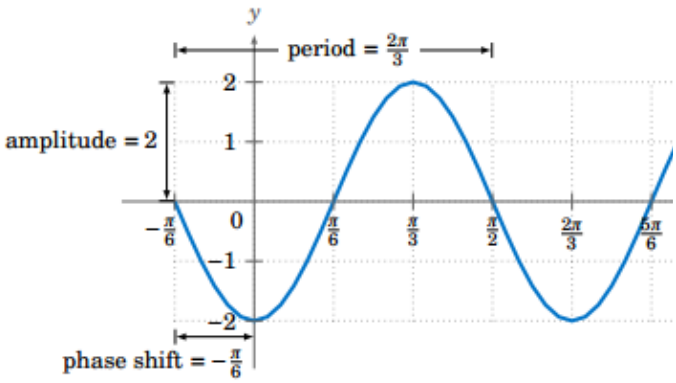


Figure 5.2.13 $y = -2\sin\left(3x + \frac{\pi}{2}\right)$

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TRIGONOMETRIC EQUATIONS

An equation involving trigonometric functions is called a trigonometric equation. for example, an equation like

$$\tan A = 0.75$$

is a trigonometric equation. We have till now, only been interested in finding a single solution. (A quadrant 1 solution between $0°$and $90°$.

We will now look at finding general solutions and developing an understanding that due to the cyclic nature of trigonometric functions we could have multiple solutions depending on the domain set.

To see what this means, take the above equation, $\tan A = 0.75$, using $\tan^{-1} 0.75$ function on your calculator (in degree mode) we get $A = 36.87°$. However we know that the tangent function has period π rad which is $180°$, that is it repeats itself every $180°$. So there are many answers for the value A, namely $36.87° + 180°$, $36.87° - 180°$, $36.87° + 360°$, $36.87° - 360°$, ect. We write this in more compact form:

$$A = 36.87° + 180°k \text{ for } k = 0, \pm 1, \pm 2 \ldots$$

or we could write this in radians as:

$$A = 0.6435 + \pi k \text{ for } k = 0, \pm 1, \pm 2 \ldots$$

EXAMPLE 5

Solve the equation $2\cos^2\theta - 1 = 0$.

Solution: Isolating $\cos^2\theta$ gives us

$$\cos^2\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \pm\frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4},$$

and since the period of cosine is 2π, we would add $2\pi k$ to each of those angles to get the general solution. But notice that the above angles differ by multiples of $\frac{\pi}{2}$. So since every multiple of 2π is also a multiple of $\frac{\pi}{2}$, we can combine those four separate answers into one:

$$\boxed{\theta = \frac{\pi}{4} + \frac{\pi}{2}k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

EXAMPLE6

Solve the equation $2\sec\theta = 1$.

Solution: Isolating $\sec\theta$ gives us

$$\sec\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \frac{1}{\sec\theta} = 2\,,$$

which is impossible. Thus, there is $\boxed{\text{no solution}}$.

EXAMPLE 7

Solve the equation $\sin\theta = \tan\theta$.

Solution: Trying the same method as in the previous example, we get

$$\begin{aligned}
\sin\theta &= \tan\theta \\
\sin\theta &= \frac{\sin\theta}{\cos\theta} \\
\sin\theta\cos\theta &= \sin\theta \\
\sin\theta\cos\theta - \sin\theta &= 0 \\
\sin\theta(\cos\theta - 1) &= 0 \\
&\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = 1 \\
&\Rightarrow \quad \theta = 0\,,\ \pi \quad \text{or} \quad \theta = 0 \\
&\Rightarrow \quad \theta = 0\,,\ \pi\,,
\end{aligned}$$

plus multiples of 2π. So since the above angles are multiples of π, and every multiple of 2π is a multiple of π, we can combine the two answers into one for the general solution:

$$\boxed{\theta = \pi k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

EXAMPLE 8

Solve the equation $\sin 2\theta = \sin\theta$.

Solution: Here we use the double-angle formula for sine:

$$\begin{aligned}
\sin 2\theta &= \sin\theta \\
2\sin\theta\cos\theta &= \sin\theta \\
\sin\theta(2\cos\theta - 1) &= 0 \\
&\Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{2} \\
&\Rightarrow \quad \theta = 0\,,\ \pi \quad \text{or} \quad \theta = \pm\frac{\pi}{3} \\
&\Rightarrow \quad \boxed{\theta = \pi k \quad \text{and} \quad \pm\frac{\pi}{3} + 2\pi k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots
\end{aligned}$$

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QUADRATIC TRIGONOMETRY

Watch these:

http://youtu.be/Hj3pBcf_ZfA

http://youtu.be/N6C8TP26K7E

http://youtu.be/p58aYq2MGaI

APPLICATIONS

There are countless numbers of applications for trigonometric models.

Take a look at this applet on springs: http://academic.sun.ac.za/mathed/trig/spring.htm

Or this one on pendulums: http://academic.sun.ac.za/mathed/trig/Pendulum.htm

Watch this movie on trigonometric functions occuring in guitar, piano and drum music: http://youtu.be/QXjdGBZQvLc

A math interlude: http://youtu.be/dbeK1fg1Rew

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MORE MODELLING QUESTIONS: EXERCISE 1.1

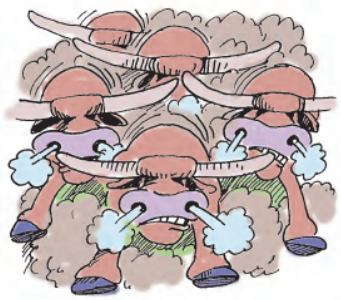
1 The population of grasshoppers after t weeks where $0 \leqslant t \leqslant 12$ is estimated by $P(t) = 7500 + 3000\sin\left(\frac{\pi t}{8}\right)$.

- **a** What is: **i** the initial estimate **ii** the estimate after 5 weeks?
- **b** What is the greatest population size over this interval and when does it occur?
- **c** When is the population **i** 9000 **ii** 6000?
- **d** During what time interval(s) does the population size exceed 10 000?

2 The model for the height of a light on a Ferris wheel is $H(t) = 20 - 19\sin\left(\frac{2\pi t}{3}\right)$, where H is the height in metres above the ground, and t is in minutes.

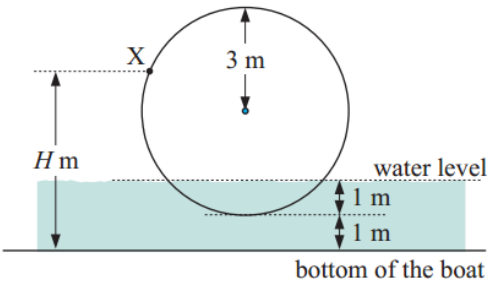
- **a** Where is the light at time $t = 0$?
- **b** At what time is the light at its lowest in the first revolution of the wheel?
- **c** How long does the wheel take to complete one revolution?
- **d** Sketch the graph of the function $H(t)$ over one revolution.

3 The population of water buffalo is given by $P(t) = 400 + 250\sin\left(\frac{\pi t}{2}\right)$ where t is the number of years since the first estimate was made.



- **a** What was the initial estimate?
- **b** What was the population size after:
  - **i** 6 months **ii** two years?
- **c** Find $P(1)$. What is the significance of this value?
- **d** Find the smallest population size and when it first occurs.
- **e** Find the first time when the herd exceeded 500.

4 A paint spot X lies on the outer rim of the wheel of a paddle-steamer. The wheel has radius 3 m and as it rotates at a constant rate, X is seen entering the water every 4 seconds. H is the distance of X above the bottom of the boat. At time $t = 0$, X is at its highest point.



- **a** Find a cosine model for H in the form $H(t) = A\cos B(t - C) + D$.
- **b** At what time does X first enter the water?

5 Over a 28 day period, the cost per litre of petrol was modelled by $C(t) = 9.2\sin\frac{\pi}{7}(t - 4) + 107.8$ cents L^{-1}.

- **a** True or false?
  - **i** "The cost per litre oscillates about 107.8 cents with maximum price \$1.17."
  - **ii** "Every 14 days, the cycle repeats itself."
- **b** What was the cost of petrol at day 7?
- **c** On what days was the petrol priced at \$1.10 per litre?
- **d** What was the minimum cost per litre and when did it occur?

TRIGONOMETRIC IDENTITIES

PYTHAGOREAN IDENTITIES

Consider again the unit circle...

It has centre (0,0) and hence equation $x^2 + y^2 = 1$

Equating that $cos\theta = x$ and $sin\theta = y$ we can then generate our first identity.

$$x^2 + y^2 = 1$$

$$\cos\theta^2 + \sin\theta^2 = 1$$

NB: See how confusing this notation is! We can't tell by looking at it if the theta is squared or if the the whole $cos\theta$ is squared. Becuase of this we use the following notation to indicate the whole trig expression is squared.

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

To develop our second Pythagorean identity we divide all terms by $\cos^2\theta$.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$1 + \frac{\sin^2\theta}{\cos^2\theta} = \sec^2\theta$$

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

To develop our third Pythagorean identities, we divide the first equation through by $\sin^2\theta$.

$$\cos^2\theta + \sin^2\theta = 1$$

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\frac{\cos^2\theta}{\sin^2\theta} + 1 = \frac{1}{\sin^2\theta}$$

$$\cot^2\theta + 1 = \frac{1}{\sin^2\theta}$$

$$\boxed{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

Suggestions...

1. **Learn well** the formulas given above (or at least, know how to find them quickly). The better you know the basic identities, the easier it will be to recognise what is going on in the problems.
2. Work on the **most complex** side and simplify it so that it has the same form as the simplest side.
3. Don't **assume** the identity to **prove** the identity. This means don't work on both sides of the equals side and try to meet in the middle. Start on one side and make it look like the other side.
4. Many of these come out quite easily if you express everything on the most complex side in terms of **sine** and **cosine** only.
5. In most examples where you see power 2 (that is, ^{2}), it will involve using the identity $\sin^2\vartheta + \cos^2\vartheta = 1$ (or one of the other 2 formulas that we derived above).

Using these suggestions, you can simplify and prove expressions involving trigonometric identities.

EXERCISE 1.2

1. Prove that $\frac{\tan y}{\sin y} = \sec y$

2. Prove that $\sin y + \sin y \cot^2 y = \csc y$

3. Prove that $\sin x \cos x \tan x = 1 - \cos^2 x$

4. Prove that $\tan x + \cot x = \sec x \csc x$

5. Prove that $\frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}$

EXERCISE 1.3

1. Prove a. $\cos\theta \tan\theta = \sin\theta$
   b. $\cos x(\csc x + \tan x) = \cot x + \sin x$
   c. $\frac{\cos^4 t - \sin^4 t}{\sin^2 t} = \cot^2 t - 1$

for answers to these 3 questions go here: http://youtu.be/9uoKutwuCio

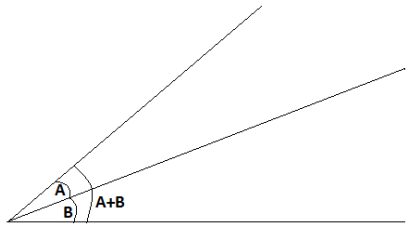
COMPOUND ANGLES

$$\sin(A+B) = \cos A \sin B + \sin A \cos B$$

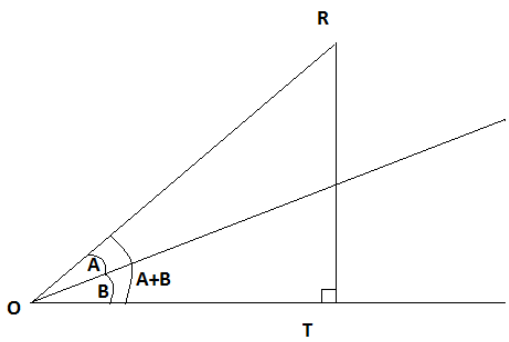
GEOMETRICAL PROOF

A compound angle is made by adding two other angles together.

Obviously in this diagram.... Angle A + B = Angle A + Angle B



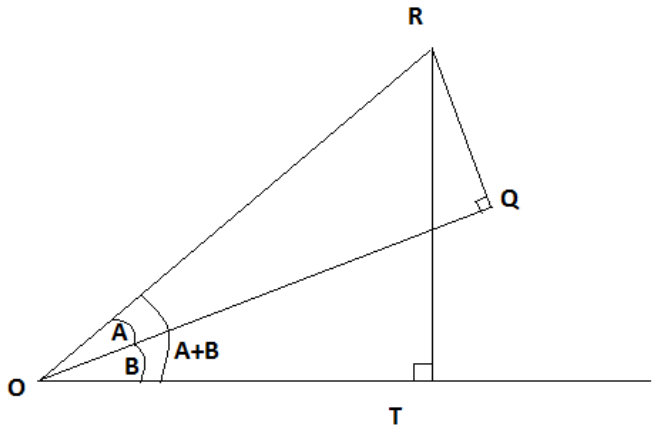
Using $\sin(A+B) = \frac{O}{H}$



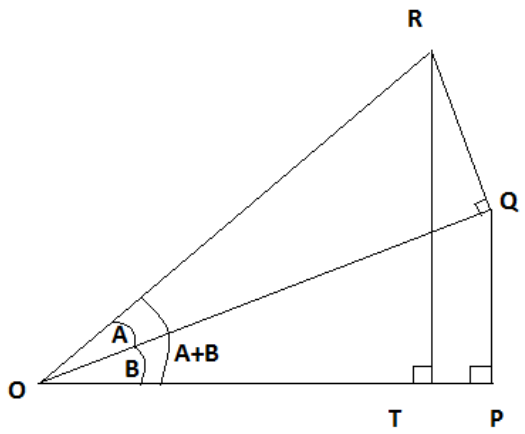
$$\sin(A+B) = \frac{O}{H}$$

$$\sin(A+B) = \frac{RT}{OR}$$

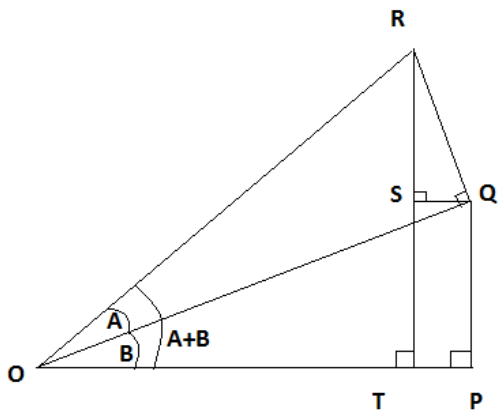
Construct a point Q so that OQR is 90°



Drop a perpendicular from point Q to line P



Construct point S such that it lies on line RT and is perpendicular to Q.



$$\sin(A+B) = \frac{RT}{OR}$$

$$\sin(A+B) = \frac{RS+ST}{OR}$$

as QP = ST

$$\sin(A+B) = \frac{RS+QP}{OR}$$

split the fraction

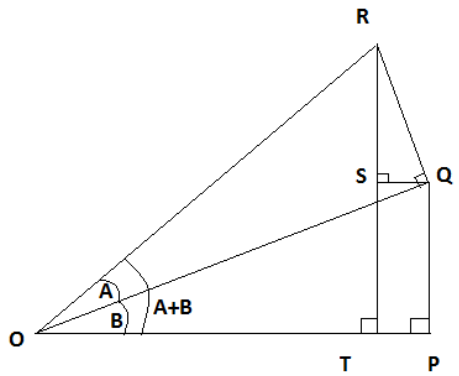
$$\sin(A+B) = \frac{RS}{OR} + \frac{QP}{OR}$$

this special construction doesn't change the equation as we are effectively only multiplying by 1.

$$\sin(A+B) = \frac{RS}{OR} \times \frac{QR}{QR} + \frac{QP}{OR} \times \frac{OQ}{OQ}$$

Now we rearrange:

$$\sin(A+B) = \frac{QR}{OR} \times \frac{RS}{QR} + \frac{OQ}{OR} \times \frac{QP}{OQ}$$



We can now replace :

$$\frac{QR}{OR} = \sin A$$

$$\frac{OQ}{OR} = \cos A$$

$$\frac{QP}{OQ} = \sin B$$

$$\frac{RS}{QR} = \cos B$$

(this last one takes a little rearranging and to use alternate angle theorem)

$$\sin(A+B) = \underbrace{\frac{QR}{OR}}_{\sin A} \times \underbrace{\frac{RS}{QR}}_{\cos B} + \underbrace{\frac{OQ}{OR}}_{\cos A} \times \underbrace{\frac{QP}{OQ}}_{\sin B}$$

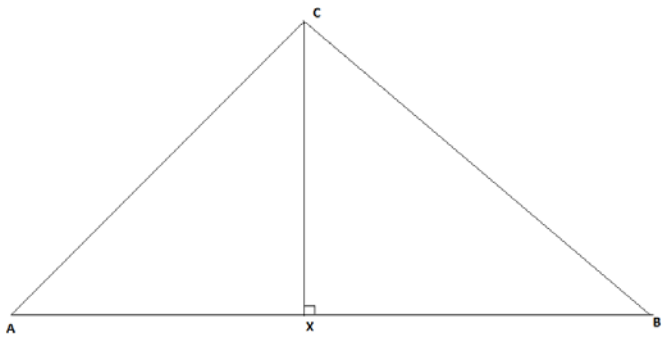
Resulting in :

$$\sin(A+B) = \cos A \sin B + \sin A \cos B$$

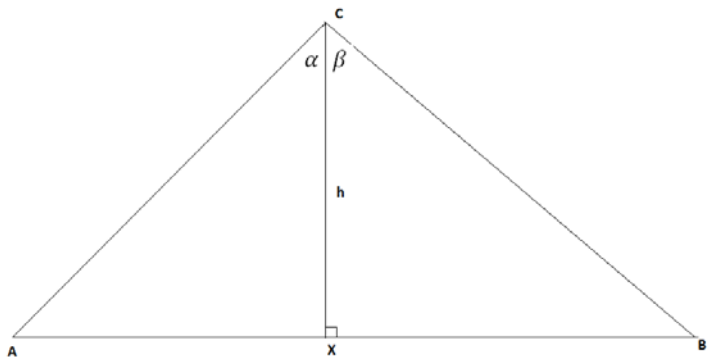
AREA PROOF

Here is another proof of the compound angle formula, using areas.

Construct a triangle ABC, with CX perpendicular to AB.



Line CX divides angle C into to angles α and β



The area of triangle ABC (using the sine rule) is

$$A = \frac{1}{2}absin\ (\alpha + \beta)$$

The area of the two smaller triangles ACX and XBC respectively are:

$$\Delta ACX = \tfrac{1}{2}bh\sin\alpha \text{ and } \Delta XBC = \tfrac{1}{2}ah\sin\beta$$

Using right angle trig, we can also see that

$$h = b\cos\alpha = a\cos\beta$$

The area ACB = area ACX + area XBC so this results in :

$$\frac{1}{2}absin\ (\alpha + \beta) = \frac{1}{2}bh\sin\alpha + \frac{1}{2}ah\sin\beta$$

$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}b\sin\alpha\, h + \frac{1}{2}a\sin\beta\, h$$

replace this with $a\cos\beta$

replace this with $b\cos\alpha$

So this yields:

$$\frac{1}{2}absin\ (\alpha + \beta) = \frac{1}{2}ab\sin\alpha\cos\beta + \frac{1}{2}ab\sin\beta\cos\alpha$$

divide through by $\frac{1}{2}ab$

$$sin\ (\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

OTHER COMPOUND ANGLES

The cosine double angle formula is: $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$

Derivations: http://youtu.be/25hYJG2_M7s and http://youtu.be/SMVExXdtXAI

EXERCISE 1.4

By working with some trigonometric algebra, and knowledge of angles in any quadrant and the signs of sine, cosine and tangent in Quadrants 1-4, solve these using the following hints:

1) Starting with

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

replace β with - β to derive

$$sin\ (\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

2) Starting with

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$$

replace β with - β to derive

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\beta\sin\alpha$$

3) Using

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

and

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$$

derive

$$\tan(\alpha+\beta)$$

4) Similarly derive

$$\tan(\alpha-\beta)$$

EXAMPLE 9

Example of using the compound angle formula to derive exact values for any angle

Find an exact value for $\tan 285°$

$$\begin{aligned}\tan 285° &= \tan(240° + 45°)\\ &= \frac{\tan 240° + \tan 45°}{1 - \tan 240° \tan 45°}\\ &= \frac{\sqrt{3}+1}{1-(\sqrt{3})(1)}\\ &= -2-\sqrt{3}\end{aligned}$$

EXERCISE 1.5

1) Describe how you would convince a friend that $\sin(x+y) \neq \sin x + \sin y$

2) Write an interpretation of the identity $\sin(90° - A) = \cos A$, using a right angled triangle.

3) Use sum or difference identities to find the exact value of each trigonometric function.

   a. $\cos 165°$

   b. $\tan \frac{\pi}{12}$

   c. $\sec\ 795°$

4) Find each exact value if $0 \leq x \leq \frac{\pi}{2}$ and $0 \leq y \leq \frac{\pi}{2}$

   a. $\sin(x-y)$ if $\sin x = \frac{4}{9}$ and $\sin y = \frac{1}{4}$

   b. $\tan(x+y)$ if $\csc\ x = \frac{5}{3}$ and $\cos y = \frac{5}{13}$

For the following use compound formulas to find exact values for the following:

5. $\sin 165°$
6. $\cos \frac{7\pi}{12}$
7. $\sin \frac{\pi}{12}$
8. $\tan 195°$
9. $\cos\left(-\frac{\pi}{12}\right)$
10. $\tan 165°$
11. $\tan \frac{23\pi}{12}$
12. $\sin 735°$
13. $\sec 1275°$
14. $\csc \frac{5\pi}{12}$
15. $\cot \frac{113\pi}{12}$

DOUBLE ANGLE FORMULA

We can use the compound formulae we have developed, to find expressions for double angles.

$$\begin{aligned}\sin 2\theta &= \sin(\theta + \theta)\\ &= \sin\theta\cos\theta + \sin\theta\cos\theta\\ &= 2\sin\theta\cos\theta\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos(\theta + \theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \cos^2\theta - \sin^2\theta\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \tan(\theta + \theta)\\ &= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta}\\ &= \frac{2\tan\theta}{1 - \tan^2\theta}\end{aligned}$$

Using the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$ and the double angle rule for $\cos 2\theta = \cos^2\theta - \sin^2\theta$, rearrange and come up with 2 more double angle formula.

Using

$$\sin^2\theta + \cos^2\theta = 1$$

and rearranging we get both:

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\cos^2\theta = 1 - \sin^2\theta$$

Such that:

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= \cos^2\theta - (1 - \cos^2\theta)\\ &= 2\cos^2\theta - 1\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta\\ &= 1 - \sin^2\theta - \sin^2\theta\\ &= 1 - 2\sin^2\theta\end{aligned}$$

EXAMPLE 10

Find the solutions to the following equation in the interval $[0,2\pi)$

$$\sin 2x + \cos x = 0$$

$\sin 2x + \cos x = 0$	Write the equation
$2\sin x \cos x + \cos x = 0$	Interchange $\sin 2x$ with the double angle formula
$\cos x\,(2\sin x + 1) = 0$	Factorise
$\cos x = 0$ or $\sin x = -\frac{1}{2}$	Use null factor law to solve
$x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{6}$	Find solutions in the interval

EXAMPLE 11

If $\sin\theta = \frac{2}{3}$ and θ has its terminal side in the first quadrant, find the exact value of $\sin 2\theta$

To use the double angle rule to evaluate $\sin 2\theta$ we need to first find $\cos\theta$

$\sin^2\theta + \cos^2\theta = 1$	
$\left(\frac{2}{3}\right)^2 + \cos^2\theta = 1$	because $\sin\theta = \frac{2}{3}$
$\cos^2\theta = \frac{5}{9}$	$1 - \frac{4}{9} = \frac{5}{9}$
$\cos\theta = \frac{\sqrt{5}}{3}$	Take square root of both sides

Now we can find $\sin 2\theta$

$$\sin 2\theta = 2\sin\theta\cos\theta$$ Double angle formula

$$= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right)$$ Substitute values for sin and cos

$$= \frac{4\sqrt{5}}{9}$$ evaluate

We can solve two of the forms of the identity for $\cos 2\theta$ for $\cos\theta$ and $\sin\theta$, respectively and the following equations result.

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \xrightarrow{\text{Solve for } \cos\theta.} \quad \cos\theta = \pm\sqrt{\frac{1+\cos 2\theta}{2}}$$

$$\cos 2\theta = 1 - 2\sin^2\theta \quad \xrightarrow{\text{Solve for } \sin\theta.} \quad \sin\theta = \pm\sqrt{\frac{1-\cos 2\theta}{2}}$$

Solving some problems using double angles: http://youtu.be/7Eo-fuy0f7g, http://youtu.be/rF36a8K_3QM,

POWER REDUCING FORMULA

We can also rearrange some of the other double angle formula to create what are called Power-Reducing Formulas.

Power-Reducing Formulas

$$\begin{aligned}\sin^2 u &= \frac{1-\cos 2u}{2}\\ \cos^2 u &= \frac{1+\cos 2u}{2}\\ \tan^2 u &= \frac{1-\cos 2u}{1+\cos 2u}\end{aligned}$$

EXAMPLE 12

Use power-reducing formulas to rewrite the following expression in terms of the first power of cosine.

$$\sin^2 x \ \cos^2 x$$

$$\begin{aligned}\sin^2 x\cos^2 x &= \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\\ &= \frac{1}{4}(1-\cos 2x)(1+\cos 2x)\\ &= \frac{1}{4}(1-\cos^2 2x)\\ &= \frac{1}{4}\sin^2 2x\\ &= \frac{1}{4}\left(\frac{1-\cos 4x}{2}\right)\\ &= \frac{1}{8}(1-\cos 4x)\end{aligned}$$

Examples: http://youtu.be/4bP1FuASneY , http://youtu.be/5Ipor4q0Jd8

EXAMPLE 13

Rewrite $\sin^4 x$as a sum of first powers of the cosines of multiple angles

$$\sin^4 x = (\sin^2 x)^2$$ Property of exponents

$$= \left(\frac{1-\cos 2x}{2}\right)^2$$ Power reducing formula

$$= \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$$ Expand

$$= \frac{1}{4}\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right)$$ Power reducing formula

$$= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{1}{8}\cos 4x$$ Distributive property

$$= \frac{1}{8}(3 - 4\cos 2x + \cos 4x)$$ Factor our common factor

FURTHER FORMULA

By using a combination of all the identities and formula so far, it is possible to write each trigonometric function in terms of the other 5. The following table demonstrates the results.

Each trigonometric function in terms of the other five.[1]

in terms of	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\csc\theta$	$\sec\theta$	$\cot\theta$
$\sin\theta =$	$\sin\theta$	$\pm\sqrt{1-\cos^2\theta}$	$\pm\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\frac{1}{\csc\theta}$	$\pm\frac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\pm\frac{1}{\sqrt{1+\cot^2\theta}}$
$\cos\theta =$	$\pm\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\pm\frac{1}{\sqrt{1+\tan^2\theta}}$	$\pm\frac{\sqrt{\csc^2\theta-1}}{\csc\theta}$	$\frac{1}{\sec\theta}$	$\pm\frac{\cot\theta}{\sqrt{1+\cot^2\theta}}$
$\tan\theta =$	$\pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\pm\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\pm\frac{1}{\sqrt{\csc^2\theta-1}}$	$\pm\sqrt{\sec^2\theta-1}$	$\frac{1}{\cot\theta}$
$\csc\theta =$	$\frac{1}{\sin\theta}$	$\pm\frac{1}{\sqrt{1-\cos^2\theta}}$	$\pm\frac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\csc\theta$	$\pm\frac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\pm\sqrt{1+\cot^2\theta}$
$\sec\theta =$	$\pm\frac{1}{\sqrt{1-\sin^2\theta}}$	$\frac{1}{\cos\theta}$	$\pm\sqrt{1+\tan^2\theta}$	$\pm\frac{\csc\theta}{\sqrt{\csc^2\theta-1}}$	$\sec\theta$	$\pm\frac{\sqrt{1+\cot^2\theta}}{\cot\theta}$
$\cot\theta =$	$\pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\pm\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\frac{1}{\tan\theta}$	$\pm\sqrt{\csc^2\theta-1}$	$\pm\frac{1}{\sqrt{\sec^2\theta-1}}$	$\cot\theta$

(from wikipedia)

HALF ANGLE FORMULA

By replacing θ with $\frac{\theta}{2}$ we can also develop half angle formula.

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{2}}$$
$$\cos\frac{u}{2} = \pm\sqrt{\frac{1+\cos u}{2}}$$
$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u} = \frac{\sin u}{1+\cos u}$$

The signs of $\sin\frac{u}{2}$ and $\cos\frac{u}{2}$ depend on the quadrant in which $\frac{u}{2}$ falls.

EXERCISE 1.6

Derive all the half angle formula for sin, cos and tan.

EXAMPLE 14

Use the half angle formula to determine the exact values of

a) $\sin 165°$

$$\sin 165° = \sin\left(\frac{330}{2}\right)^{\circ} = \sqrt{\frac{1-\cos 330°}{2}} = \sqrt{\frac{1-\sqrt{3}/2}{2}}$$

b) $\cos 165°$

$$\cos 165° = \cos\left(\frac{330}{2}\right)^{\circ} = -\sqrt{\frac{1+\cos 330°}{2}} = -\sqrt{\frac{1+\sqrt{3}/2}{2}}$$

c) $\tan 165°$

$$\tan 165° = -\sqrt{\frac{1-\sqrt{3}/2}{1+\sqrt{3}/2}} = -\sqrt{\frac{2-\sqrt{3}}{2+\sqrt{3}}}$$

EXAMPLE 15

Find the exact value of $\sin 105°$

Being by noting that $105°$is half of $210°$. Then using the half angle formula for sine, and the fact that $105°$ lies in quadrant 2, (to determine the sign), we get...

$$\begin{aligned}\sin 105° &= \sqrt{\frac{1 - \cos 210°}{2}} \\ &= \sqrt{\frac{1 - (-\cos 30°)}{2}} \\ &= \sqrt{\frac{1 + (\sqrt{3}/2)}{2}} \\ &= \frac{\sqrt{2 + \sqrt{3}}}{2}.\end{aligned}$$

The positive square root is chosen because sin is positive in Quadrant 2.

Some more video examples: http://youtu.be/uFbbF-lYFjM, http://youtu.be/Q48HHoLauyg, http://youtu.be/FFXaeJYaGVY

EXERCISE 1.7

1) Tamika calculated the exact value of $\sin 15°$ in two different ways. Using the difference identity for sine, $\sin 15°$ was $\frac{\sqrt{6}-\sqrt{2}}{4}$. When she used the half angle identify $\sin 15°$ equaled $\frac{\sqrt{2-\sqrt{3}}}{2}$. Which answer is correct? Explain.

2) Use a half angle identity to find the exact value for

a) $\sin\frac{\pi}{8}$

b) $\tan 165°$

c) $\cos\frac{7\pi}{12}$

3) Verify the following identities

a) $1+\frac{1}{2}\sin 2A = \frac{\sec A+\sin A}{\sec A}$

b) $\sin\frac{x}{2}\cos\frac{x}{2} = \frac{\sin x}{2}$

4) Consider an AC circuit consisting of a power supply and a resistor. If the current in the circuit at time t, is $I_0 \sin \omega t$, then the power delivered to the resistor is $P = I_0^2 R \sin^2 \omega t$, where R is the resistance. Express the power in terms of $\cos 2\omega t$

5) Use the half angle identity to find the exact value for:

a) $\cos 15°$

b) $\sin 75°$

c) $\tan\frac{5\pi}{12}$

d) $\sin\frac{3\pi}{8}$

e) $\cos\frac{7\pi}{12}$

6) If θ is an angle in the first quadrant, and $\cos\theta = \frac{1}{4}$, find $\tan\frac{\theta}{2}$

PRODUCT TO SUM FORMULAS

Are formulas that take the products of trigonometric functions and write them as sums of other trigonometric functions.

$\sin(u)\cos(v) = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

How to Derive:

We start with $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$ and $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$.

Line them up and add both sides. Then,

$$\begin{aligned} \sin(u+v) &= \sin(u)\cos(v) + \cos(u)\sin(v) \\ \sin(u-v) &= \sin(u)\cos(v) - \cos(u)\sin(v) \\ \hline \sin(u+v) + \sin(u-v) &= 2\sin(u)\cos(v) \end{aligned}$$

$$\frac{\sin(u+v) + \sin(u-v)}{2} = \sin(u)\cos(v)$$

$$\sin(u)\cos(v) = \tfrac{1}{2}\left[\sin(u+v) + \sin(u-v)\right]$$

$\cos(u)\sin(v) = \frac{1}{2}[\sin(u+v) - \sin(u-v)]$

How to Derive:

We start with $\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$ and $\sin(u-v) = \sin(u)\cos(v) - \cos(u)\sin(v)$.

Line them up and subtract both sides. Then,

$$\begin{aligned} \sin(u+v) &= \sin(u)\cos(v) + \cos(u)\sin(v) \\ \sin(u-v) &= \sin(u)\cos(v) - \cos(u)\sin(v) \\ \hline \sin(u+v) - \sin(u-v) &= 2\cos(u)\sin(v) \end{aligned}$$

$$\frac{\sin(u+v) - \sin(u-v)}{2} = \cos(u)\sin(v)$$

$$\cos(u)\sin(v) = \tfrac{1}{2}\left[\sin(u+v) - \sin(u-v)\right]$$

$\cos(u)\cos(v) = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$

How to Derive:

We start with $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$ and $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$.

Line them up and add both sides. Then,

$$\begin{aligned} \cos(u-v) &= \cos(u)\cos(v) + \sin(u)\sin(v) \\ \cos(u+v) &= \cos(u)\cos(v) - \sin(u)\sin(v) \\ \hline \cos(u-v) + \cos(u+v) &= 2\cos(u)\cos(v) \end{aligned}$$

$$\frac{\cos(u-v) + \cos(u+v)}{2} = \cos(u)\cos(v)$$

$$\cos(u)\cos(v) = \tfrac{1}{2}\left[\cos(u-v) + \cos(u+v)\right]$$

$\sin(u)\sin(v) = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$

How to Derive:

We start with $\cos(u-v) = \cos(u)\cos(v) + \sin(u)\sin(v)$ and $\cos(u+v) = \cos(u)\cos(v) - \sin(u)\sin(v)$.

Line them up and subtract both sides. Then,

$$\begin{aligned} \cos(u-v) &= \cos(u)\cos(v) + \sin(u)\sin(v) \\ \cos(u+v) &= \cos(u)\cos(v) - \sin(u)\sin(v) \\ \hline \cos(u-v) - \cos(u+v) &= 2\sin(u)\sin(v) \end{aligned}$$

$$\frac{\cos(u-v) - \cos(u+v)}{2} = \sin(u)\sin(v)$$

$$\sin(u)\sin(v) = \tfrac{1}{2}\left[\cos(u-v) - \cos(u+v)\right]$$

Product-to-Sum Formulas

$$\begin{aligned} \sin u \sin v &= \frac{1}{2}[\cos(u-v) - \cos(u+v)] \\ \cos u \cos v &= \frac{1}{2}[\cos(u-v) + \cos(u+v)] \\ \sin u \cos v &= \frac{1}{2}[\sin(u+v) + \sin(u-v)] \\ \cos u \sin v &= \frac{1}{2}[\sin(u+v) - \sin(u-v)] \end{aligned}$$

EXAMPLE 16

Use the product-to-sum formulas to write the following as a sum or difference.

$$\begin{aligned} \sin\frac{\pi}{4}\cos\frac{\pi}{12} &= \frac{1}{2}\left[\sin\left(\frac{\pi}{4} + \frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4} - \frac{\pi}{12}\right)\right] \\ &= \frac{1}{2}\left[\sin\frac{\pi}{3} + \sin\frac{\pi}{6}\right] \\ &= \frac{1}{2}\left[\frac{\sqrt{3}}{2} + \frac{1}{2}\right] \\ &= \frac{\sqrt{3}+1}{4} \end{aligned}$$

EXAMPLE 17

Rewrite the product $\cos 5x \sin 4x$ as a sum or difference.

Using the appropriate product-to-sum formula, we can get:

$$\begin{aligned}\cos 5x \sin 4x &= \frac{1}{2}[\sin(5x+4x) - \sin(5x-4x)] \\ &= \frac{1}{2}\sin 9x - \frac{1}{2}\sin x\end{aligned}$$

SUM TO PRODUCT FORMULAS

Sometimes it's just useful to go the other way!

Sum-to-Product Formulas

$$\begin{aligned}\sin u + \sin v &= 2\sin\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \sin u - \sin v &= 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \cos u + \cos v &= 2\cos\left(\frac{u+v}{2}\right)\cos\left(\frac{u-v}{2}\right) \\ \cos u - \cos v &= -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)\end{aligned}$$

EXERCISE 1.8

In Problems 1–4, write each product as a sum or difference involving sine and cosine.

1. $\sin 3m \cos m$

2. $\cos 7A \cos 5A$

3. $\sin u \sin 3u$

4. $\cos 2\theta \sin 3\theta$

In Problems 5–8, write each difference or sum as a product involving sines and cosines.

5. $\sin 3t + \sin t$

6. $\cos 7\theta + \cos 5\theta$

7. $\cos 5w - \cos 9w$

8. $\sin u - \sin 5u$

Evaluate Problems 9–12 exactly using an appropriate identity.

9. $\sin 195° \cos 75°$

10. $\cos 75° \sin 15°$

11. $\cos 15° \cos 75°$

12. $\sin 105° \sin 165°$

Evaluate Problems 13–16 exactly using an appropriate identity.

13. $\cos 285° + \cos 195°$

14. $\sin 195° + \sin 105°$

15. $\cos 15° - \cos 105°$

16. $\sin 75° - \sin 165°$

FORMULAS

Think that's a lot of trig formulas. Here is a link to a helpful summary sheet.

http://www.ecalc.com/pics/math/pdf/trigonometry_laws_identities.pdf