WEEK 17 RATIONAL FUNCTIONS

A rational number is one that can be represented by a fraction.

A rational function is basically a division of two polynomial functions. That is, it is a polynomial divided by another polynomial. In formal notation, a rational function would be symbolized like this:

$$f(x) = \frac{s(x)}{t(x)}$$

Why can't $t(x) = 0$?

Where $s(x)$ and $t(x)$ are polynomial functions, and $t(x)$ cannot equal zero.

Here is an example of a rational function:

$$f(x) = \frac{x^2 + x - 20}{x^2 - 3x - 18}$$

To understand the behaviour of a rational function it is very useful to see its polynomials in factored form.

The polynomials in the numerator and the denominator of the above function would factor like this:

$$f(x) = \frac{(x+5)(x-4)}{(x+3)(x-6)}$$

THE DOMAIN (DEFINED BY THE DENOMINATOR)

Now the roots of the denominator are obviously x = -3 and x = 6. That is, if x takes on either of these two values, the denominator becomes equal to zero. Since we can't divide by zero, the function is not defined for these two values of x.

We say that the function is **discontinuous** at x = -3 and x = 6. It can never be equal to these values – i.e. the function does not continue from the left to the right through those x-values.

Other values for x do not cause the function to become undefined, so, we say that the function is continuous at all other values for x. In other words, all real numbers except -3 and 6 are allowed as inputs to this function. The domain for the function, therefore, as expressed in interval notation is:

$$(-\infty,\ -3) \cup (-3,\ 6) \cup (6,\ +\infty)$$

THE X-INTERCEPTS (DEFINED BY THE NUMERATOR)

The x-intercept happens when y=0.

A rational function can be considered a fraction, and a fraction is equal to zero when the numerator is equal to zero.

For our rational function example this happens when the polynomial in the numerator is equal to zero, and this will happen at the roots of the numerator polynomial. The roots of the numerator polynomial are x = -5 and x = 4. That is, when x takes on either of these two values the numerator becomes zero, and the output of the function, or y-value, also becomes zero.

So, the x-intercepts for this rational function are x = -5 and x = 4. Notice that the function is defined at these two values. (It is only not defined at x = -3 and x = 6.) That makes these true x-intercepts. If the function was not defined at x = 4 because 4 was a root of the denominator polynomial, (which it is not in our example here), then x = 4 would not be an x-intercept even though it made the numerator equal to zero. You cannot have an x-intercept for a function at a point where the function does not exist!

THE Y-INTERCEPT (FOUND WHEN X=0)

What about the y-intercept/s? Well, they happen when $x = 0$. If we look at our first un-factored form for this function, expressed in 'y =' form, we have:

$$y = \frac{x^2 + x - 20}{x^2 - 3x - 18}$$

Now, setting x = 0 we get:

$$y = \frac{(0)^2 + (0) - 20}{(0)^2 - 3(0) - 18}$$

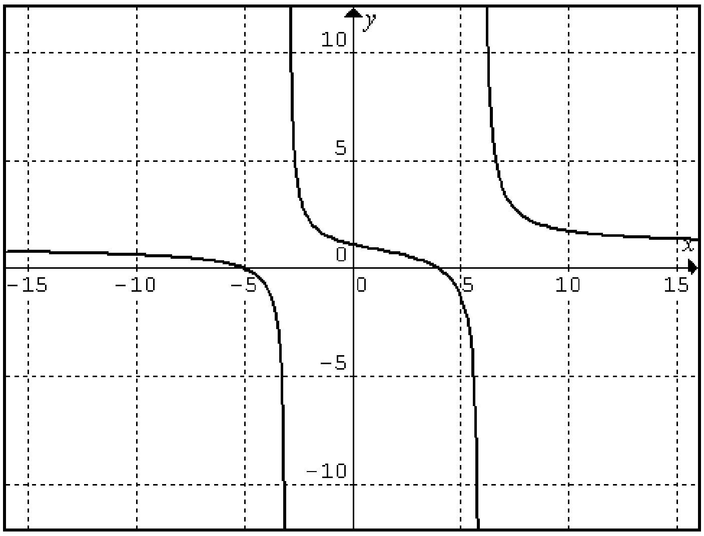
$$y = \frac{-20}{-18}$$

$$y = \frac{10}{9}$$

$$y \approx 1.11$$

That is, the graph crosses the y-axis at $y = \frac{10}{9}$ (about 1.11).

THE GRAPH



- The function line is discontinuous, or 'breaks', at x = -3 and x = 6. That is, if you were drawing the graph by hand, you would have to lift the pen off the paper at x = -3 and x = 6. That is what we mean by a discontinuity. Notice that these locations for the discontinuities, (x = -3 and x = 6) are the roots of the polynomial in the denominator.
- The function crosses the x-axis at x = -5 and x = 4. These are the same as the values which we calculated above for the x-intercepts.
- The function crosses the y-axis just a bit above 1, at about 1.1. This is the same location as the calculated y-intercept above.

VERTICAL AND HORIZONTAL ASYMPTOTES

Rational functions often have vertical or horizontal asymptotes, (or both). The key to sketching rational functions is to identify these key points.

HORIZONTAL ASYMPTOTES

These can often be found by looking at the degree of the numerator (n) and comparing it to the degree of the denominator (d).

- if $n > d$ then there is no horizontal asymptote
- if $n = d$ then the graph will have an horizontal asymptote on the line $y = \frac{a}{b}$ (where a is the leading coefficient of the numerator, and b is the leading coefficient of the denominator)
- If $n < d$ then the line $y = 0$ is a horizontal asymptote

VERTICAL ASYMPTOTES

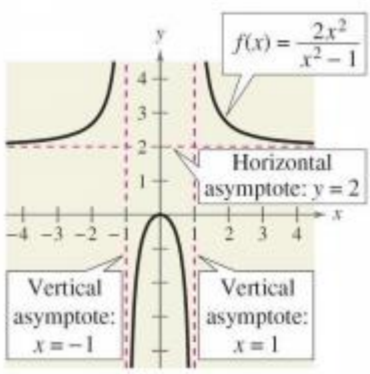
These can often be found by looking at the denominator only. Identify where the discontinuities lie (find the zeros of the denominator polynomial).

EXAMPLE 1

$$f(x) = \frac{2x^2}{x^2 - 1}$$

Vertical asymptotes happen when $x^2 - 1 = 0$, which is at $x = \pm 1$, these are the 2 vertical asymptotes.

The degree of the numerator (n) is 2, the degree of the denominator (d) is 2, so $n = d$, so the line $y = \frac{a}{b} = \frac{2}{1}$ which is the line $y = 2$ is a *horizontal asymptote*.



EXAMPLE 2

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}$$

Before we look too much further, we need to factorise this one...

$$f(x) = \frac{(x-1)(x+2)}{(x+2)(x-3)}$$

$$= \frac{(x-1)\cancel{(x+2)}}{\cancel{(x+2)}(x-3)}$$

$$= \frac{(x-1)}{(x-3)}, x \neq -2$$

Vertical asymptotes happen when $x - 3 = 0$, which is $x = 3$ this is the vertical asymptote.

The degree of the numerator (n) is 2, the degree of the denominator (d) is 2, so $n = d$, so the line $y = \frac{a}{b} = \frac{1}{1}$ which is the line $y = 1$ is a *horizontal asymptote*.

OBLIQUE (OR SLANT) ASYMPTOTES

Not only can some rational functions have horizontal and vertical asymptotes, but some also have slant or oblique asymptotes. If the degree of the numerator (n), is exactly one more than the degree of the denominator (d), then the graph will have an oblique asymptote.

To find an oblique asymptote, it is necessary to divide the numerator by the denominator. You can use polynomial long division, or the CAS. (on a CAS use propFrac command)

Finding slant asymptotes: http://youtu.be/--vh9zgZZmQ

EXAMPLE 3

$$f(x) = \frac{x^2 - x}{x + 1}$$

Horizontal asymptote - none

Vertical asymptote - where x+1=0, which is at x=-1

Oblique asymptote - found by using propFrac command.... $f(x) = x - 2 + \frac{2}{x+1}$

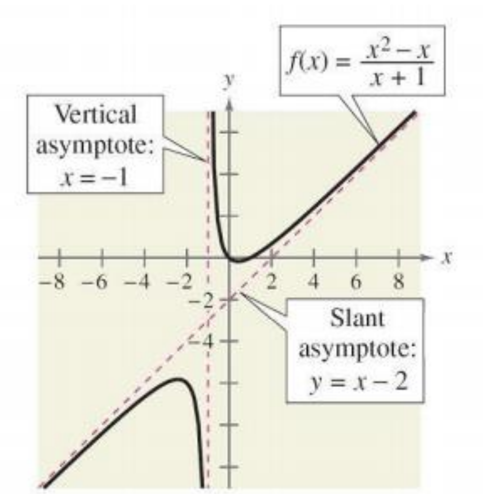
We now consider what happens as x gets very large.....

$$f(x) = x - 2 + \frac{2}{x+1}$$

Gets very large

Gets very small and disappears to 0

Leaving us with the end result of y=x-2 being our oblique asymptote.



SKETCHING RATIONAL FUNCTIONS

The following is a list of things that need consideration when trying to sketch a rational function

a) horizontal asymptotes
b) vertical asymptotes
c) oblique asymptotes (extension)
d) x=intercepts
e) y=intercepts
f) plot points in each of the segmented domains to get location
g) check/graph on CAS to support your sketch

Graphing rational functions: http://youtu.be/hWjMovgqvi4

http://youtu.be/IbsLYHzKoH0

EXERCISES 2.1 RATIONAL FUNCTIONS

For questions 1-10, (a) find the domain of the function, (b) identify any horizontal and vertical asymptotes and (c) create a sketch.

1. $f(x) = \frac{1}{x^2}$

2. $f(x) = \frac{3}{(x-2)^3}$

3. $f(x) = \frac{2+x}{2-x}$

4. $f(x) = \frac{1-5x}{1+2x}$

5. $f(x) = \frac{x^2+2x}{2x^2-x}$

6. $f(x) = \frac{3x^2+1}{x^2+x+9}$

7. $f(x) = \frac{x^2-25}{x^2+5x}$

8. $f(x) = \frac{3x^2+x-5}{x^2+1}$

9. $f(x) = \frac{x-3}{|x|}$

10. $f(x) = \frac{x+1}{|x|+1}$

11. A utility company burns coal to generate electricity. The cost C (in dollars) of removing p% of the smokestack pollutants is given by $C = \frac{80000p}{100-p}$ for $0 \leq p < 100$. You are a member of state parliament considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal.

a) How much additional cost would the utility company incur as a result of the new law?
b) Why is the domain state for p<100?
c) Where the asymptotes in this function, and what are their real life meaning?

12. The cost C (in millions of dollars) of removing p% of the industrial and municipal pollutants discharged into a river is given by: $C = \frac{255p}{100-p}$ $0 \leq p < 100$.

a) Find the cost of removing 10% of the pollutants
b) find the cost of removing 40% of the pollutants
c) Find the cost of removing 75% of the pollutants
d) Graph the function on your CAS, explain how you get the correct viewing window.
e) According to this model, would it be possible to remove 100% of the pollutants? Explain?

13. The game commission in South Africa introduces 100 deer into newly acquired state game lands. The population N of the herd is given by $N = \frac{20(5+3t)}{1+0.04t}$, $t \geq 0$, where t is the time in years.

a) graph the model
b) find the population when $t = 5, t = 10$ and $t = 25$
c) What is the limiting size of the herd as time increases? Explain.

14. Determine whether the following two statements are true of false, justify your answer.

a) A rational function can have infinitely many vertical asymptotes
b) $f(x) = x^3 - 2x^2 - 5x + 6$ is a rational function

In questions 15-18 write a rational function f that has the specified characteristics. (there are many correct answers)

15. Vertical asymptotes, x=-2 and x=1

16. Vertical asymptote: None
Horizontal asymptote: y=0

17. Vertical asymptote: None
Horizontal asymptote: y=2

18. Vertical asymptote: x=0 and x=5/2
Horizontal asymptote: y=-3