COMPLEX NUMBERS COURSE NOTES

HAWKER COLLEGE
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CONTENTS

Activities aim at being a connection between theory and practice, they will involve generalisations from mini investigations

Challenges are more difficult or conceptually challenging problems

Practice and Consolidation sets are in built question sets, or references to the text book questions that apply to that topic.

WEEK 10

Start with a story.

http://mathforum.org/johnandbetty/

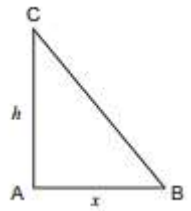
WEEK 11

A LITTLE HISTORY

The history of complex numbers can be dated back as far as the ancient Greeks. When solving polynomials, they decided that no number existed that could solve $x^2 = -1$. Diophantus of Alexandria (AD 210 – 294 approx) tried to solve the following problem.


DIOPHANTI
ALEXANDRINI
ARITHMETICORVM
LIBRI SEX.
ET DE NVMERIS MVLTANGVLIS
LIBER VNVS.
AVCTORE CLAVDIO GASPARE BACHETO
LVTETIAE PARISIORVM,
Sumptibus SEBASTIANI CRAMOISY, via Iacobea, sub Ciconiis.
M. DC. XXI.
CVM PRIVILEGIO REGIS

"Find the sides of a right-angled triangle of perimeter 12 units and area 7 squared units"



Letting$AB = x$, $AC = h$ then

$area = \frac{1}{2}xh$ and

$$perimeter \ = \ x + h + \sqrt{x^2 + h^2}$$

ACTIVITY 1

Show that the two equations above can be reduced to

$$6x^2 - 43x + 84 = 0$$

when the perimeter =12 and area = 7. Does this have real solutions?

Another mathematician Jerome Caran (1501 – 1576) also worked on a similar problem. He tried to solve the problem of finding two numbers, a and b, whose sum is 10 and whose product is 40.

$$a + b = 10$$

$$ab = 40$$

Eliminating b gives

$$a(10 - a) = 40$$

$$a^2 - 10a + 40 = 0$$

Solving this quadratic would give the solution

$$a = \frac{1}{2}\left(10 \pm \sqrt{-60}\right) = 5 \pm \sqrt{-15}$$

From this we can see that there are no real solutions, (solutions that are part of the real number system), but if we can continue to use these numbers then $a = 5 + \sqrt{-15}, b = 5 - \sqrt{-15}$ would satisfy the original conditions.

We can say that these are solutions to the original problem but they are not real numbers.

It wasn't until the nineteenth century that these solutions could be fully understood.

DEFINITIONS

Complex numbers are often denoted by z.

Complex numbers are built on the concept of being able to define the square root of negative one.

$$\text{Let } i^2 = -1$$

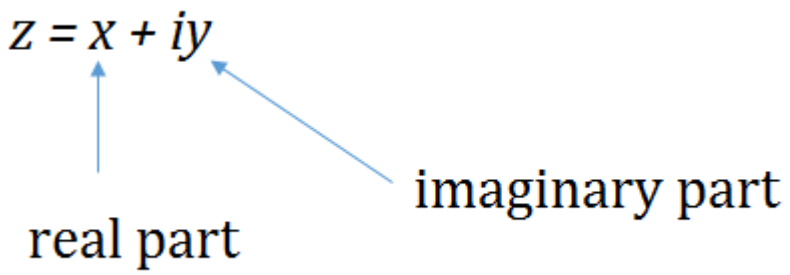
$$\therefore i = \sqrt{-1}$$

Just like how $\mathbb{R}$ denotes the real number system, (the set of all real numbers) we use $\mathbb{C}$ to denote the set of complex numbers.

$$z = x + iy \ \in \mathbb{C}, \text{for some } x, y \ \in \mathbb{R}$$

Read as → $z = x + iy$ which is an element of the set of complex numbers where x and y are real numbers.

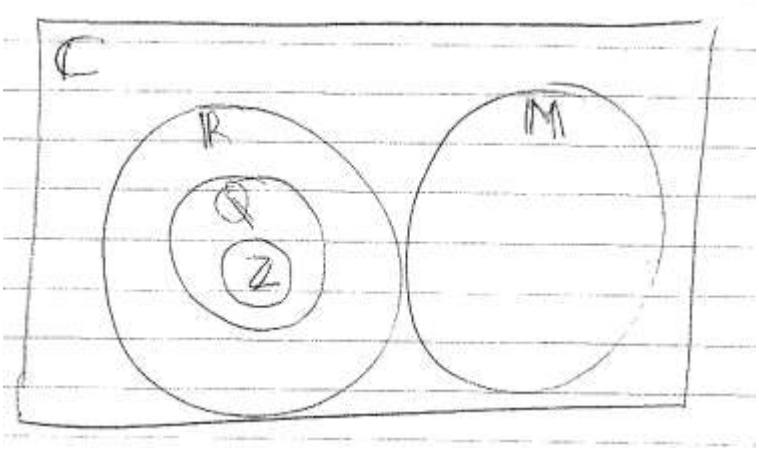
So a number like $5 + 3i$ is a complex number.



The real part of z *is* $\mathbb{R}(z) = \mathrm{Re}(z) = x$

The imaginary part of z *is* $\Im(z) = \mathrm{Im}(z) = y$

Every real number x can be written as $x + 0i$, so set of real numbers is a subset of the set of complex numbers.



Complex Numbers

Real Numbers

Imaginary Numbers

Integers

Rational Numbers

Natural Numbers

So for $5 + 3i$

$$\mathbb{R}(z) = \mathrm{Re}(z) = 5$$

$$\Im(z) = \mathrm{Im}(z) = 3$$

So what…. Why do we even need to construct $\sqrt{-1}$?

One thing complex numbers enables us to do is solve polynomial equations that would have had NO REAL SOLUTIONS.

WATCH THIS

http://youtu.be/VTAsbx5wBaI and http://youtu.be/tbWfxuV-IeE Rewriting powers of I, by PatrickJMT

http://youtu.be/8PT_ZN8YxVY How to write surds (radicals) using complex notation, by PatrickJMT

PLAY with the applet on this page http://www.picomonster.com/imaginary-numbers-lesson-1/

Play with the applet on this page http://www.geogebratube.org/student/m35847

EXAMPLE 1

To solve $x^2 - 2x + 3 = 0$

We can use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$

$$x = \frac{2 \pm \sqrt{4 - 12}}{2}$$

$$x = \frac{2 \pm \sqrt{-8}}{2}$$

$$x = \frac{2 \pm 2\sqrt{-2}}{2}$$

$$x = 1 \pm \sqrt{-2}$$

$$x = 1 \pm \sqrt{(-1)(2)}$$

$$x = 1 \pm \sqrt{-1}\,\sqrt{2}$$

$$x = 1 \pm \sqrt{2}\,i$$

Or we can complete the square

$$x^2 - 2x + 3 = 0$$

$$x^2 - 2x + (-1)^2 + 3 - (-1)^2 = 0$$

$$(x - 1)^2 + 2 = 0$$

$$(x - 1)^2 = -2$$

$$(x - 1) = \pm\sqrt{-2}$$

$$x = 1 \pm \sqrt{-2}$$

$$x = 1 \pm \sqrt{2}\,i$$

PRACTICE AND CONSOLIDATION 1

Solve the following equations

a) $x^2 - 1 = 0$

b) $x^2 + x + 1 = 0$

c) $x^2 - x - 6 = 0$

d) $x^2 + 1 = 0$

e) $x^2 - 2x - 2 = 0$

f) $2x - 7 = 4x^2$

g) $x^2 - 2x + 2 = 0$

h) $3x^2 - 4x + 2 = 0$

CHALLENGE 1

Show the algebraic process for completing the square of

$$ax^2 + bx + c = 0$$

Then state the roots and possible cases.

CHALLENGE 2

Graph both these functions, where are the solutions to the pair of simultaneous equations?

$$x^2 - y^2 = a$$

$$2xy = b$$

CHALLENGE 3

Patterns of Complex Powers

$$i, i^2, i^3, i^4, \ldots$$

Make a list of powers of i up to 15. Simplify the results. Then generalise this pattern.

PRACTICE AND CONSOLIDATION 2

Maths Quest Specialist Mathematics Chapter 3A Questions 5, 6, 8, 9

COMPLEX ARITHMETIC

ADDITION AND SUBTRACTION

Complex numbers can be added and subtracted

Add/subtract corresponding real and imaginary parts

WATCH THIS

http://youtu.be/UU4F8hwnM7s Adding and Subtracting Complex Numbers

Play with this applet - http://www.geogebratube.org/student/m3426

EXAMPLE 2

$$(3+7i)+(2+i)=(2+3)+(7+1)i=5+8\,i$$

EXAMPLE 3

$$(9-2i)-(-2+6i)=9-2i+2-6i=11-8i$$

PRACTICE AND CONSOLIDATION 3

Maths Quest Advanced General Chapter 1D 3 from each question 1-5

Maths Quest Specialist Mathematics Chapter 3B Question 2

MULTIPLICATION

Complex numbers can be multiplied, just remember that every $i^2=-1$

EXAMPLE 4

$$\begin{aligned}(1-3i)(4+2i)&=(1)(4)-(3i)(4)+(1)(2i)-(3i)(2i)\\&=4-12i+2i-6i^2\\&=4-10i+6\\&=10-10i\end{aligned}$$

EXAMPLE 5

$$(4-3i)(4+3i) = 16 - 9i^2 = 16 + 9 = 25$$

In this example, see how when we multiply two complex numbers that are identical except that the sign is different we get a real number. We call these numbers conjugates of each other.

PRACTICE AND CONSOLIDATION 4

Simplify the following

a) $(3+7i)+(18-5i)$

b) $(15-6i)-(3-9i)$

c) $8(6-9i)+i(4+7i)$

d) $(2+7i)(4-6i)$

e) $(6+13i)(4+2i)$

f) $(3+2i)^2$

g) i^3

h) i^4

i) $(3+i)^2+(3-i)^2$

j) $(1-i)^3$

k) $(2+i)^4+(2-i)^4$

l) $(a+ib)(a-ib)$

CHALLENGE 4

We know surd laws such as

$$\sqrt{a}\times\sqrt{b} = \sqrt{ab}$$
$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$
$$\sqrt{a}\pm\sqrt{b} \text{ has no simplification}$$
$$x\sqrt{a}+y\sqrt{a} = (x+y)\sqrt{a}$$

Extend these laws to see what happens if a&b are < 0. (not complex just negative)

CONJUGATES (A PROCESS FOR DIVISION)

If $z = x + iy$ then $\bar{z}$ (pronounced zed bar), is given by $= x - iy$, and this is called the *complex conjugate* of z.

So for $z = 4 + 3i$ then $\bar{z} = 4 - 3i$

Just like with dealing with surds, we can also rationalist the denominator, when dealing with complex numbers. This effectively gives us a process for division.

EXAMPLE 6

$$\begin{aligned}\frac{3+6i}{2-3i} &= \frac{3+6i}{2-3i} \times \frac{2+3i}{2+3i} \\ &= \frac{(3+6i)(2+3i)}{(2-3i)(2+3i)} \\ &= \frac{6+12i+9i+18i^2}{4+9} \\ &= \frac{-12+21i}{13}\end{aligned}$$

WATCH THIS

http://youtu.be/KPSj4-76eEc Multiplying and Dividing Complex Numbers, by PatrickJMT

PatrickJMT also has some worked examples

Multiplying #1 http://youtu.be/ROVuhl8wo_g

Multiplying #2 http://youtu.be/nyCTm9Ff0E4

Dividing #1 http://youtu.be/JoLq9NaLOvI

Dividing #2 http://youtu.be/9I4QsSV1XDg

Dividing #3 http://youtu.be/ZvZ_dLlDS0U

PRACTICE AND CONSOLIDATION 5

Simplify to the form $(a + bi)$

a) $\frac{4}{i}$

b) $\frac{1-i}{1+i}$

c) $\frac{4+5i}{6-5i}$

d) $\frac{4i}{(1+2i)^2}$

e) $\frac{3}{6+i} + \frac{2}{3-2i}$

CHALLENGE 5

Verify which of the following hold true, check with $z = 1 + i$ and $w = 2 - 2i$

then prove for $z = a + bi$ and $w = c + di$.

a) $\overline{z - w} = \bar{z} - \bar{w}$
b) $\overline{zw} = \bar{z} \times \bar{w}$
c) $\overline{z^n} = (z)^n$ use (b) to help here
d) $\overline{\left[\frac{z}{w}\right]} = \frac{\bar{z}}{\bar{w}}$
e) What is $z + \bar{z}$? Generalise
f) What is $z - \bar{z}$? Generalise

PRACTICE AND CONSOLIDATION 6

Maths Quest Advanced General Chapter 1E Questions 1-6

Maths Quest Specialist Mathematics Chapter 3B Question 4, 5, 8, 9, 12

Chapter 3C Question 4, 6, 8, 9, 10, 17, 20

COMPLEX EQUATIONS

If two complex numbers are equal then the real and imaginary parts are also equal. We call this equating like parts.

EXAMPLE 7

If $a + 6i = 3 + 6i$, then $a = 3$

If $8 - bi = 8 + 7i$, then $b = -7$

We can use this process to solve algebraic problems involving complex numbers

EXAMPLE 8

Find x, y if $(3 + 2i)^2 - 3(x + iy) = x + iy$

$$\begin{aligned} LHS &= (3 + 2i)^2 - 3(x + iy) \\ &= (9 + 12i - 4) - 3x - 3iy \\ &= (5 - 3x) + (12i - 3iy) \end{aligned}$$

$$\therefore 5 - 3x = x$$

$$5 = 4x$$

$$x = \frac{5}{4}$$

&

$$\begin{aligned} 12 - 3y &= y \\ 12 &= 4y \\ y &= 3 \quad \blacksquare \end{aligned}$$

EXAMPLE 9

Find x, y if $\frac{x}{1-i} + \frac{y}{4+3i} = 2 - 4i$

$$LHS = \frac{x}{1-i} + \frac{y}{4+3i}$$

$$= \frac{x}{1-i} \times \frac{1+i}{1+i} + \frac{y}{4+3i} \times \frac{4-3i}{4-3i}$$

$$= \frac{x(1+i)}{1+1} + \frac{y(4-3i)}{16+9}$$

$$= \frac{x+xi}{2} + \frac{4y-3iy}{25}$$

Now then, this LHS needs to equal $2 - 4i$

SO

$$\frac{x+xi}{2} + \frac{4y-3iy}{25} = 2 - 4i$$

$$25(x+xi) + 2(4y-3iy) = 50(2-4i)$$

$$25x + 25xi + 8y - 6iy = 100 - 200i$$

$$(25x+8y) + i(25x-6y) = 100 - 200i$$

By equating like parts we can then get simultaneous equations to solve…

$$25x + 8y = 100 \quad (1)$$

$$25x - 6y = -200 \quad (2)$$

$$14y = 300 \quad (1) - (2)$$

$$y = \frac{300}{14}$$

$$x = -\frac{20}{7}$$

PRACTICE AND CONSOLIDATION 7

1) Find the requested pronumeral if it is $\in \mathbb{R}$

a) If $a + 6i = 8 + 6i$, then find a

b) If $-3 + 2bi = -3 + 16i$, then find b

c) If $2a + 21i = 8 + 7bi$, then find a, b

d) If $-(2a - 18i) = -4 + 18i$, then find a

2) Find x, y if

a) $(1 - 3i)^2 - 5(x + iy) = x + iy$

b) $(2 + i)^3 + (3 - i)(x - iy) = x + iy$

3) Solve the following equations for $x, y \in \mathbb{R}$

a) $3 + 5i + x - yi = 6 - 2i$

b) $x + yi = (1 - i)(2 + 8i)$

4) Determine the complex number z which satisfies

a) $z(3 + 3i) = 2 - i$

b) $z^2 - 2z + (3 - 4i) = 6 + 3i$

SUMMARY QUESTIONS 1

Question 1 Solve the following equations:

a) $x^2 + 9 = 0$
b) $9x^2 + 25 = 0$
c) $x^2 + 2x + 2 = 0$
d) $x^2 + x + 1 = 0$
e) $2x^2 + 3x + 2 = 0$

Question 2 What quadratic equation has roots $3 \pm \sqrt{6}i$?

Question 3 Simplify the following complex numbers in the form $x + yi$

a) $(3 + 2i) + (2 + 4i)$
b) $(4 - 3i) + (4 + 3i)$
c) $(4 + 3i) - (2 + 5i)$
d) $(2 + 6i) - (2 - 6i)$
e) $(4 + 6i)^2$
f) $(1 + i)(1 - i)(3 + i)$
g) $(3 + 2i)(3 - 2i)(2 + 3i)(2 - 3i)(9 - i)$

Question 4 Find the value of the real number y such that

$$(3 + 2i)(1 + iy)$$

Is (a) real (b) imaginary

Question 5 Simplify

a) i^2
b) i^3
c) i^4
d) i^5
e) $\frac{1}{i}$
f) $\frac{1}{i^2}$
g) $\frac{1}{i^3}$

Question 6 If $w = 3 - 2i$, find

a) w^2
b) $\frac{1}{w}$
c) $\frac{1}{w^2}$
d) $\overline{w}$
e) $\overline{w^3}$

Question 7 Write in the form $x + yi$

a) $\frac{2+3i}{1+2i}$

b) $\frac{-3+6i}{-2-3i}$

c) $\frac{9i}{3-7i}$

d) $\frac{5}{7+2i}$

e) $\frac{4-2i}{i}$

f) $\frac{p+qi}{r+si}$

Question 8 Solve for z when

a) $z(2+i) = 3 - 2i$

b) $(z+i)(1-i) = 2 + 3i$

c) $\frac{1}{z} + \frac{1}{2-i} = \frac{3}{1+i}$

Question 9 Find the values of the real numbers x and y that satisfy

$$\frac{2}{2-i} + \frac{yi}{i+3} = \frac{2}{1+i}$$

Question 10 Given that p and q are real and that $1 + 2i$ is a root of the equation

$$z^2 + (p + 5i)z + q(2 - i) = 0$$

Determine:

a) The values of p and q

b) The other root of the equation

Question 11 The complex numbers u, v and w are related by

$$\frac{1}{u} = \frac{1}{v} + \frac{1}{w}$$

Given that $v = 3 + 4i$, $w = 4 - 3i$, find u.

WEEK 12

ARGAND DIAGRAM

Argand diagram is what we call the plane that will allow us to plot complex numbers. It is named after the Swiss mathematician Jean Argand (1768 – 1822).

Using the x-axis as the real axis, and the y-axis as the imaginary axis, the ordered pairs (a, b) reflect complex numbers of the form $a + bi$.

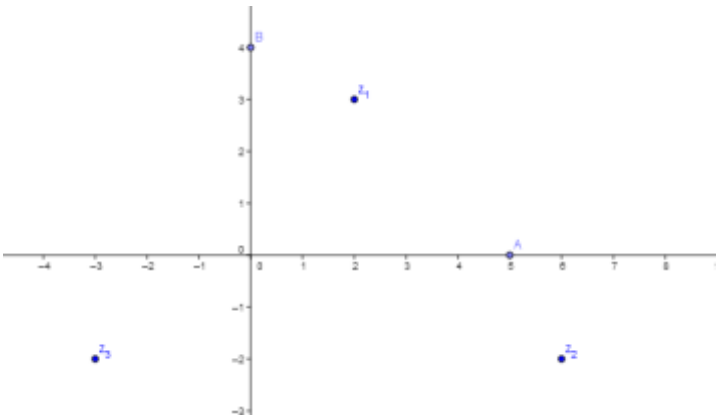
WATCH THIS

http://youtu.be/UU4F8hwnM7s Complex Numbers Graphing, adding and subtracting by PatrickJMT

EXAMPLE 11

Plot the following on the Argand diagram.

$$z_1 = 2 + 3i;\ z_2 = 6 - 2i;\ z_3 = -3 - 2i;\ A = 5;\ B = 4i$$



CHALLENGE 6

If $z = 1 - i$, find $z, z^2, z^3, z^4, z^5, z^6$. Plot these points on an Argand Diagram.

Is there a geometric pattern?

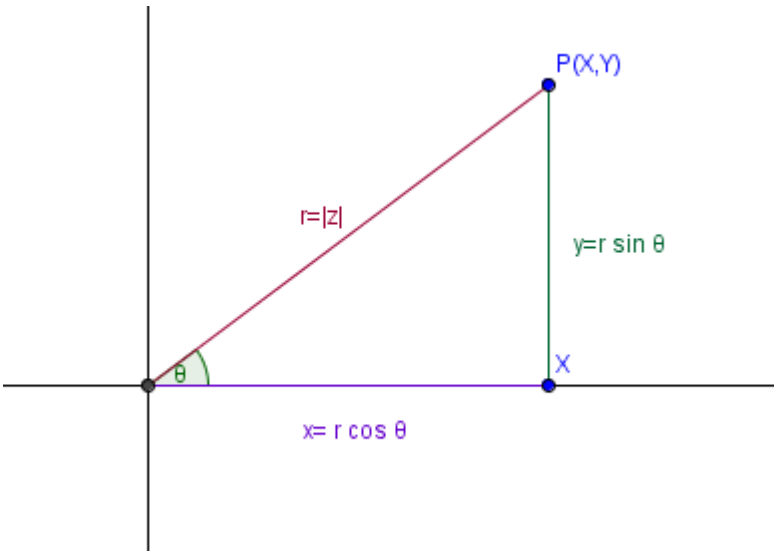
Can you generalise your result?

PRACTICE AND CONSOLIDATION 8

Let $w = 3 + 6i, u = -2 + 4i, v = -3 - 2i, p = 5 - i$

Plot the complex numbers u, v, p and w on an Argand diagram and label them

Now let's look at just one point in detail. $P(x, y) = x + iy$



$$z = x + iy$$

$$= r\cos\theta + i\,r\sin\theta$$

$$= r\,(\cos\theta + i\sin\theta)$$

This is directly related to the unit circle work you completed last year, if you cannot see the connection, come and see me or revise that section of work.

PRACTICE AND CONSOLIDATION 9

Maths Quest Advanced General Chapter 1F Questions 3-5

COMPLEX COMPONENTS

MODULUS (DISTANCE FROM THE ORIGIN TO THE POINT P)

Modulus is denoted using many different notations: the letter r, $mod\ z$, $|z|$, and $|x + iy|$.

The modulus is the SIZE of the number, it is measured as a scalar distance from the origin to the complex point (x, y), from $x + iy$. Using basic trigonometry we can see that the modulus is the distance r, which is $\sqrt{x^2 + y^2}$ (see below).

Using Pythagoras

$$r^2 = x^2 + y^2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = |z| = |x + iy| = \sqrt{x^2 + y^2}$$

WATCH THIS

Some examples on graphing and finding the modulus by PatrickJMT http://youtu.be/Eh_C1wiPuwo and http://youtu.be/znN3Qys8uS4

ARGUMENT (THE ANGLE Θ)

The argument of a complex number is the angle the point (x, y) from the complex point $x + iy$ makes with the positive x-axis. {-180®<θ<180®}

The argument is denoted using a number of different notations: θ, $arg\ z$, $arg\ (x + iy)$.

For $x \neq 0, \tan\theta = \frac{y}{x}$

The convention for complex numbers is to use radians as the measure for angles.

POLAR FORM

As we can see from the above trigonometric definitions surrounding complex numbers,

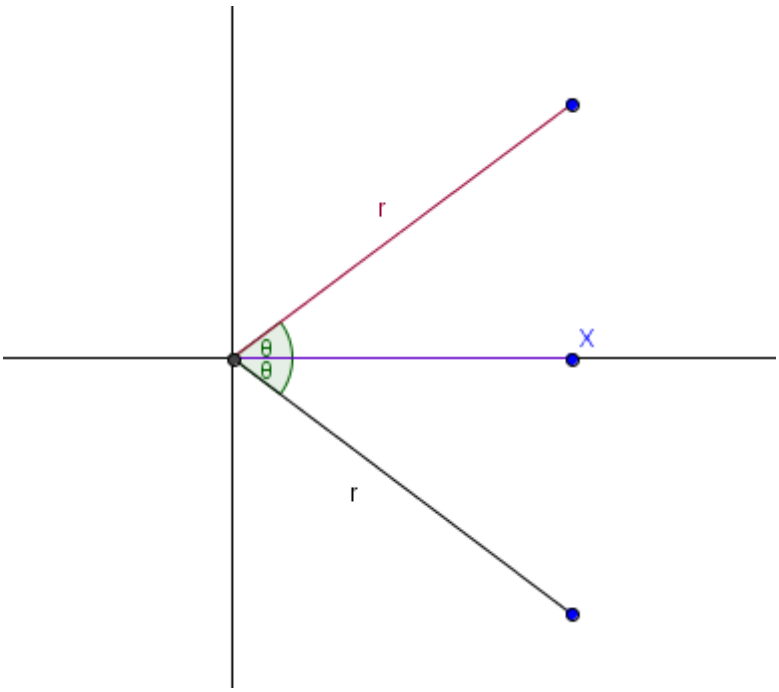
$$z = x + iy$$

$$= r\cos\theta + i\,r\sin\theta$$

$$= r\,(\cos\theta + i\sin\theta)$$

We abbreviate this to $r\ cis\ \theta$. (where cis stands for cos theta plus i sine theta)

Using polar form if $z = r\ cis\ \theta$ then $\bar{z} = r\ cis(-\theta)$



You can see here that a conjugate is a reflection of the point on the x-axis in the complex plane.

WATCH THIS

Writing a number in polar form, by PatrickJMT http://youtu.be/6z6fzPXUbSQ

Converting from Complex to Polar Form, By PatrickJMT http://youtu.be/tl56MgKA3Z8

PRACTICE AND CONSOLIDATION 10

a) Write the following numbers in polar form
   i. $6 + 3i$
   ii. $4 - i$
   iii. $-3 + 2i$
   iv. $-3 - 2i$
   v. $\sqrt{2} - 4i$

b) What is the connection between parts iii and iv

c) Write the following numbers in $a + bi$ form:
   i. $3\ cis\frac{\pi}{4}$
   ii. $5\ cis\ \pi$
   iii. $6\ cis\ \left(-\frac{2\pi}{3}\right)$
   iv. $\sqrt{3}\ cis\ \sqrt{3}$

Plot the complex numbers u, v, p and w on an Argand diagram and label them

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PRACTICE AND CONSOLIDATION 11

Maths Quest Specialist Mathematics Chapter 3D Question 3, 5, 6 and 7 (half of each), 9 and 10 (half of each)

NOTATION

Modulus argument form is yet another way to represent a complex number. $[r, \theta] = r\ cis\ \theta$.

MULTIPLICATION AND DIVISION

Using Polar or Modulus Argument Forms, multiplication can be distilled down to a very simple form.

For example, suppose that $z_1 = [r_1, \theta_1]$ and $z_2 = [r_2, \theta_2]$, then

$$z_1 z_2 = [r_1, \theta_1][r_2, \theta_2]$$

$$= r_1 \operatorname{cis} \theta_1 \quad r_2 \operatorname{cis} \theta_2$$

$$= r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)i]$$

Using what is called the double angle formula, we can simplify this to

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] = [r_1 r_2, \theta_1 + \theta_2\,]$$

(Multiply the modulus, add the arguments)

For Division, if $z_1 = r_1 \operatorname{cis}\theta_1$ and $z_2 = r_2 \operatorname{cis}\theta_2$ then $\frac{z_1}{z_2} = \frac{r_1}{r_2} cis\ (\theta_1 - \theta_2)\,)$

(Divide the modulus, subtract the arguments)

WATCH THIS

Multiplying and Dividing in Polar Form by PatrickJMT Ex 1 http://youtu.be/in_dAlpjUKk and Ex 2 http://youtu.be/QxboPshf__g

PRACTICE AND CONSOLIDATION 12

Given that $z_1 = [3, 0.7], z_2 = [2, 1.2], z_3 = [4, -0.5]$

a) Find $z_1 \times z_2$ and $z_1 \times z_3$
b) Show that $[1,0] \times z_1 = z_1$
c) (i) find a complex number $z = [r, \theta]$, such that $z \times z_2 = [1,0]$

   (ii) find a complex number $z = [r, \theta]$, such that $z \times z_3 = [1,0]$

d) For any complex number $[r, \theta]$ show that

$$\left[\frac{1}{r}, -\theta\right] \times [r, \theta] = [1,0] \qquad (r > 0)$$

PRACTICE AND CONSOLIDATION 13

Maths Quest Specialist Mathematics Chapter 3E Question 1, 4, 7, 8, 13, 16

SUMMARY QUESTIONS 2

Question 1 Mark on an Argand diagram the points representing the following:

a) 2
b) $5i$
c) i
d) $2+i$
e) $5-2i$
f) $-3+2i$

Question 2 The points, A, B, C and D represent the numbers z_1, z_2, z_3 and z_4 and 0 is the origin.

a) If OABC is a parallelogram and $z_1 = 3+2i, z_2 = 5+7i$ find z_3
b) Find z_2 and z_4 when ABCD is a square and
   i. $z_1 = 1+2i, z_3 = 7+8i$
   ii. $z_1 = 6-2i\ ,\ z_3 = 6i$

Question 3 Show that

a) $|\bar{z}| = |z|$
b) $\arg \bar{z} = \arg z$

And illustrate these results on an Argand Diagram.

Question 4 Find the modulus and argument of $z_1, z_2, z_1 z_2\ and \frac{z_1}{z_2}$ when $z_1 = 1+i$ and $z_2 = \sqrt{3}+i$.
What do you notice? Does it matter what $z_1\ or\ z_2$ are?

Question 5 Write in the form $a+bi$

a) $\left[4, \frac{\pi}{3}\right]$
b) $\left[5, \frac{\pi}{2}\right]$
c) $\left[3\sqrt{2}, -\frac{3\pi}{4}\right]$
d) $[4, 13\pi]$

Question 6 Write in polar form

a) $1+i$
b) $-2+I$
c) -6
d) $7i$
e) $3+2i$
f) $-3-2i$
g) $3-2i$
h) $-3+2i$

Question 7

a) Plot the following complex numbers on an Argand diagram and label them

$$z_1 = [4,0], z_2 = \left[3, \frac{\pi}{2}\right], z_3 = \left[2, -\frac{\pi}{2}\right], z_4 = \left[3, \frac{\pi}{3}\right], z_5 = \left[2, \frac{5\pi}{3}\right]$$

b) Let the complex number $z = \left[1, \frac{\pi}{2}\right]$. Calculate $z \times z_1, z \times z_2, etc.$. and plot the points on the same diagram as in (a). What do you notice?

c) Repeat (b) using the complex number $z = \left[1, \frac{\pi}{3}\right]$

d) In general what happens when a complex number is multiplied by $[1, \theta]$? Make up some examples to illustrate your answer.

WEEK 13

AXIOMS

Complex numbers uphold the following axioms

a. The set of Complex numbers ($\mathbb{C}$) are closed under addition – which is commutative and associative.
b. The set of Complex numbers ($\mathbb{C}$) are closed under multiplication – which is commutative and associative
c. Multiplication distributes over addition
d. Multiplicative identity is the number 1.
e. There is a complex number 0, (not equal to 1) which serves as the additive identity.

PRACTICE AND CONSOLIDATION 14

For the axioms listed above, provide examples (generic where possible) that demonstrate them.

THEOREMS

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then if $z_1 = z_2$ then $r_1 = r_2$ & if $\theta_1 = \theta_2$ (equating like parts)

$|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2 \pm 2\pi$

WATCH THIS

Conjugate Pair Theorem – by PatrickJMT http://youtu.be/ZQjq-o9ZBek

PRACTICE AND CONSOLIDATION 15

Maths Quest Specialist Mathematics Chapter 3F Question 1, 3, 4, 10, 12, 13, 14, 15, 20

PRACTICE AND CONSOLIDATION 16

For the following questions you should use your CAS to practice factorising, solve by hand and then use the CAS to double check.

a) Reduce $z^4 + 8z^2 - 9$ to linear factors and then solve
b) Factorise $3z^3 - 4z^2 + 3z - 4$ over C and then solve $3z^3 - 4z^2 + 3z - 4 = 0$
c) Factorise $z^3 - 5z^2 + 16z - 30$ over C and then solve $z^3 - 5z^2 + 16z - 30 = 0$
d) Factorise $z^3 - z^2 + z - 1$ over C and then solve $z^3 - z^2 + z - 1 = 0$
e) Factorise $z^3 - 3z^2 + z + 5$ over C and then solve $z^3 - 3z^2 + z + 5 = 0$
f) Factorise $z^2 + 4$ over C and find linear factors of $z^3 - z^2 + 4z - 4$ over C.
Then solve $z^3 - z^2 + 4z = 4$.
g) Find real values of b, for which bi is a solution of $z^4 - 2z^3 + 7z^2 - 4z + 10 = 0$. Hence find solution set.
h) Find all real values of b, for which bi is a solution of $z^4 - 2z^3 + 11z^2 - 18z + 18 = 0$. Hence find solution set.

PRACTICE AND CONSOLIDATION 17

Maths Quest Advanced General Chapter 1G Questions 1-7 (a selection of each)

WEEK 14

CHALLENGE 7

Pick any complex number, plot it on an Argand plane.

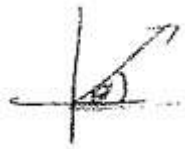
a) Multiply it repeatedly by i, plotting the result each time.
b) Multiply it repeatedly by –i plotting the result each time.

Is there a pattern? Generalise your result.

RAYS AND LINES IN THE COMPLEX PLANE

The basics of sketching in the complex plane.

$\{z: \arg(z) = \theta\}$ is a ray starting at origin with an angle of θ with the positive real axis.



$\{z: \arg(z + a + bi) = \theta\}$ is a ray with an angle of θ with the positive real axis translated from the origin a units in real direction and b units in the imaginary direction.

$\{z: Re(z) = a, a \in \mathbb{R}\}$ is a vertical line through $x = a$

$\{z: Re(z + a + bi) = c, a, b, c \in \mathbb{R}\}$ is a vertical line translated (-a) in the Real direction and (-b) in the Imaginary direction. Since this graph is a vertical line the imaginary translation is irrelevant. This is a vertical line passing through $(c - a) + 0i$

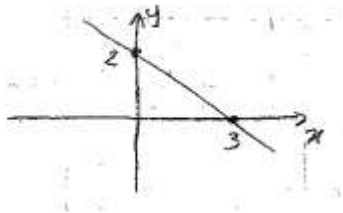
$\{z: Im(z) = b, b \in \mathbb{R}\}$ is a horizontal line through $y = b$

$\{z: Im(z + a + bi) = c, a, b, c \in \mathbb{R}\}$ is a horizontal line translated (-a) in the Real direction and (-b) in the Imaginary direction. Since this graph is a horizontal line the real component is irrelevant. This is a horizontal line passing through $0 + (c - b)i$

If $z = x + iy$, then $z + a + bi = c$ becomes $x + iy + a + bi = c$, which is $[x + a] + [y + bi] = c$. So then if we want $Re(z + a + bi) = c$ then we just look at $x + a = c$, and if we wanted the $Im(z + a + bi) = c$ then we just look at $y + b = c$.

EXAMPLE 12

Sketch $6 = Re[\,(2-3i)z]$



First we let $z = x + iy$

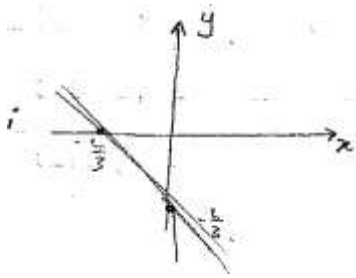
Then

$$\begin{aligned}(2-3i)z &= (2-3i)(x+iy)\\ &= 2x - 3ix + 2iy + 3y\\ &= (2x+3y) + (2x-3x)i\end{aligned}$$

So $Re[\,(2-3i)z\,] = 6$ is $2x + 3y = 6$

EXAMPLE 13

Sketch $-2 = Im\,[\,(3+2i)z\,]$



Let

$$\begin{aligned}z &= x + iy, then\ (3+2i)z\\ &= (3+2i)(x+iy)\\ &= (3x + 2ix + 3iy - 2y\\ &= (3x-2y) + (2x+3y)i\end{aligned}$$

So because we are only sketching $Im\,[\,(3+2i)z\,] = -2$ then $2x + 3y = -2$

EXAMPLE 14

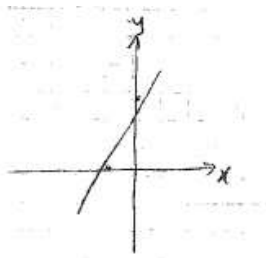
Sketch $0 = Re[\,zi\,]$

Let $z = x + iy$, then $zi = (x+iy)I = xi - y$

So because we are only sketching the $Re[\,zi] = 0$ then $y = 0$

EXAMPLE 15

Sketch $-125 = Re[\,z(6+2i)\,]$



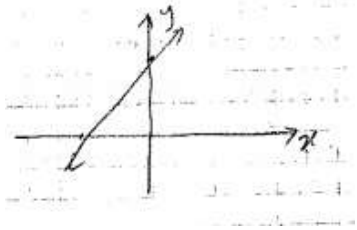
Let $z = x + iy$, then

$$\begin{aligned}z(6+2i) &= (x+iy)(6+2i)\\ &= 6x + 6iy + 2ix - 2y\\ &= (6x-2y) + (2x+6y)i\end{aligned}$$

So because we are only wanting to sketch $-125 = Re[\,z(6+2i)\,]$, then $6x - 2y = -125$

EXAMPLE 16

Sketch $12 = Im\,[(3-i)z + (2-i)z\,]$



Let $z = x + iy$, then

$$\begin{aligned}(3-i)z + (2-i)z &= z[\,(3-i) + (2-i)] \\ &= z\,[\,5 - 2i] \\ &= (x+iy)(5-2i) \\ &= 5x + 5iy - 2ix + 2y \\ &= (5x+2y) - (2x-5y)i\end{aligned}$$

So we just look at $12 = 2x - 5y$

PRACTICE AND CONSOLIDATION 18

Sketch the following graphs

a)

i. $\left\{z : arg(z) = \frac{\pi}{4}\right\}$

ii. $\left\{z : arg(z) = \frac{5\pi}{6}\right\}$

iii. $\left\{z : arg(z) = \frac{-2\pi}{3}\right\}$

iv. $\left\{z : arg(z) = \frac{-\pi}{2}\right\}$

b)

i. $\left\{z: \arg(z+3+8i) = \frac{\pi}{2}\right\}$

ii. $\left\{z: \arg(z-2+6i) = \frac{2\pi}{3}\right\}$

iii. $\left\{z: \arg(z-6-i) = \frac{-5\pi}{6}\right\}$

iv. $\left\{z: \arg(z+4-5i) = \frac{-\pi}{4}\right\}$

c)

i. $\{z: Re(z) = 3\}$

ii. $\{z: Re(z) = -3\}$

iii. $\{z: Im(z) = 6\}$

iv. $\{z: Im(z) = -2\}$

d) Sketch

i. $6 = Re[\,(2-3i)z]$
ii. $-40 = Re[\,(35+12i)z]$
iii. $-12 \;=\; Im\,[\,(4-8i)z\,]$
iv. $18 = \; Im\,[\,(-2-6i)z\,]$
v. $0 \;=\; Im[\,zi\,]$
vi. $0 \;=\; Re[\,-zi\,]$
vii. $-46 \;=\; Re[\,z(6-3i)]$
viii. $20 \;=\; Im[\,z(-2-2i)]$
ix. $111 \;=\; Re[\,z(4+2i)]$
x. $-37 \;=\; Im[\,z(8-4i)\,]$
xi. $12 \;=\; Im\,[(4-2i)z \;+\; zi\,]$
xii. $8 \;=\; Re\,[(5-i)z-(23-51i)z\,]$
xiii. $-14 \;=\; Re\,[(11-6i)z \;+\; 4i(2+i)z\,]$
xiv. $28 \;=\; Im\,[(9-i)z-\; 5(2+2i)z\,]$

CIRCLES AND ELLIPSES IN THE COMPLEX PLANE

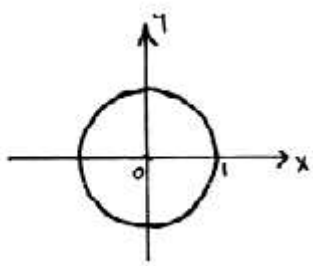
CIRCLES

Since |z| defines the size of z from the origin (modulus), then {z: |z| = r } defines the collection of points that are all distance r from the origin. (a circle)

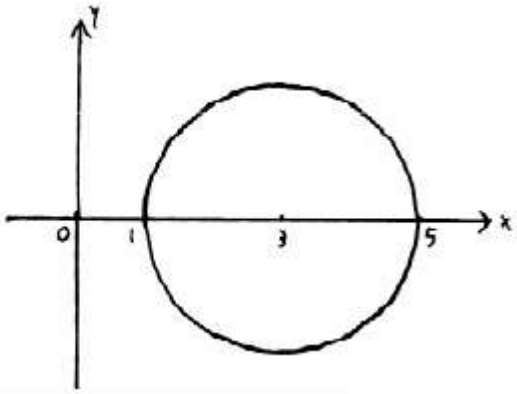
$\{z\colon |z+a+bi| \;=\; r, r, a, b \;\in R\,\}$ is a circle, radius r with centre $(-a-bi)$. The centre has been translated from the origin – a units in the real direction, and – b in the imaginary direction.

EXAMPLE 17

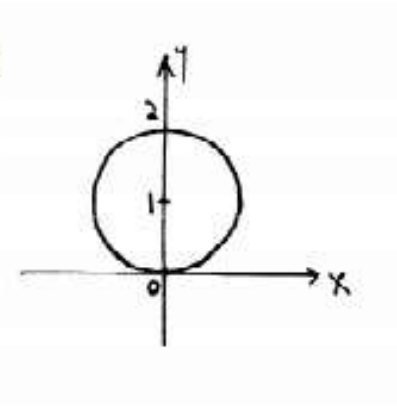
$|z| = 1.$



. $|z-3| = 2$

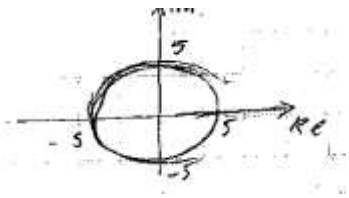


$|z-i| = 1.$



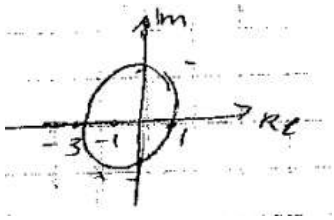
EXAMPLE 18

a) $\{z: |z| = 5\}$
Is a circle with radius 5 and centre at the origin.



b) $\{z: |z+1| = 2\}$

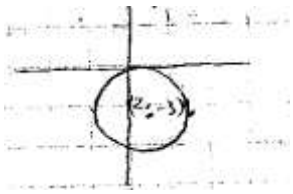
Is a circle, with radius 2, and centre (-1, 0)



EXAMPLE 19

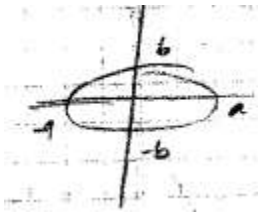
$\{z: |z-2+3i| = 3\}$

Is a circle, with radius 3 and centre $(2-3i)$



ELLIPSES

The Cartesian equation of an ellipse with centre at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Since in our complex system $z = x + iy$ and $x = Re(z)$ and $y = Im(z)$

Then $\frac{[Re(z)]^2}{a^2} + \frac{[Im(z)]^2}{b} = 1$

This results in an ellipse with centre at the origin, semi major radius of a and semi minor radius of b.

To translate an ellipse to have centre at $(h + ki)$

$$\frac{[Re(z) - h]^2}{a^2} + \frac{[Im(z) - k]^2}{b} = 1$$

PRACTICE AND CONSOLIDATION 19

Sketch the following

a)

i. $\left\{z: \frac{Re(z)^2}{16} + \frac{Im(z)^2}{25} = 1\right\}$

ii. $\left\{z: \frac{Re(z)^2}{20} + \frac{Im(z)^2}{16} = 1\right\}$

iii. $\left\{z: \frac{Re(z)^2}{121} + \frac{Im(z)^2}{81} = 1\right\}$

b)

i. $\left\{z: \frac{[Re(z)-3]^2}{16} + \frac{[Im(z)-6]^2}{25} = 1\right\}$

ii. $\left\{z: \frac{[Re(z)+14]^2}{10} + \frac{[Im(z)-17]^2}{24} = 1\right\}$

iii. $\left\{z: \frac{[Re(z)-27]^2}{36} + \frac{[Im(z)+18]^2}{4} = 1\right\}$

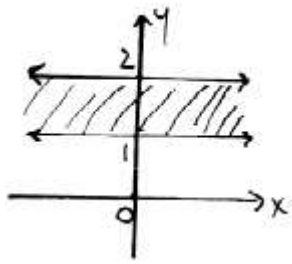
WEEK 15

REGIONS AND AREAS IN THE COMPLEX PLANE

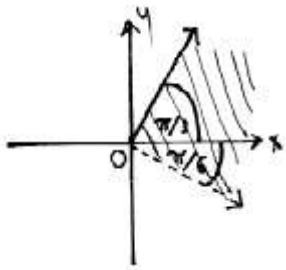
When you combine rays, lines, circles and ellipses with inequalities you end up with shaded regions and areas in the complex plane. With regions, if less than that it is inside, if it is greater than it is outside. To combine regions and areas, AND and OR are used, these are mathematically referred to as UNION (∪) and INTERSECTION. (∩)

EXAMPLES 20 WITH RAYS AND LINES

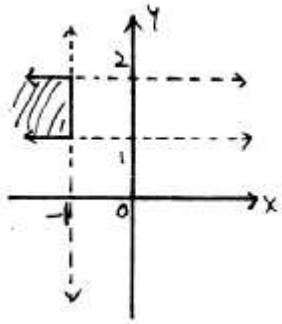
Sketch $1 \leq Im(z) \leq 2$



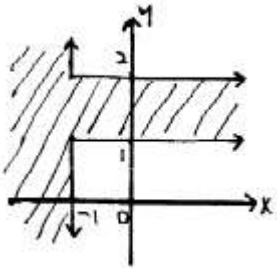
Sketch $-\frac{\pi}{6} \leq \arg z \leq \frac{\pi}{3}$



Sketch $1 < Im(z) \leq 2 \ \cap Re(z) \leq -1$

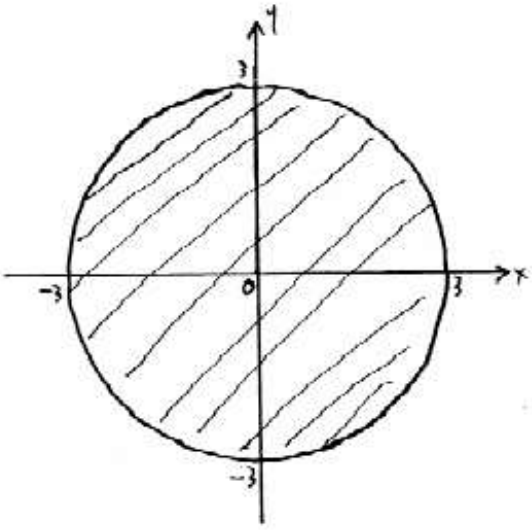


Sketch $1 \leq Im(z) \leq 2 \ \cup Re(z) \leq -1$

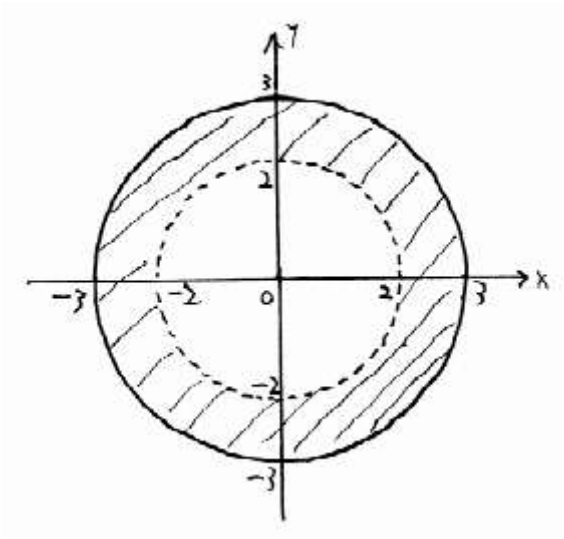


EXAMPLES 21 WITH CIRCLES

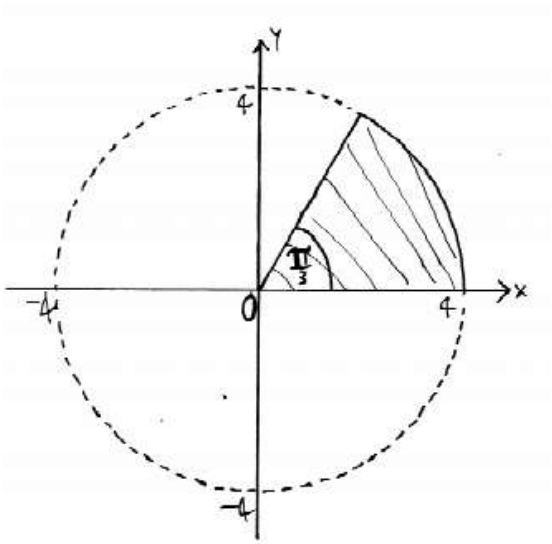
Sketch $|z| \leq 3$



Sketch $2 < |z| \leq 3$



Sketch $|z| \leq 4 \cap 0 < \arg z < \frac{\pi}{3}$



PRACTICE AND CONSOLIDATION 20

a) $\{z: |z-1| > 1 \ \cup Im(z+i) > 1\}$

b) $\{ z: 4 \leq |z+ \ -2i| \cap Im(z+2i) \geq 4 \ \cap |z+2-2i| < 6 \ \}$

c) $\{ z: |z-2| + |z+2| < 8 \ \cap 2 \leq |z+1-2i| \leq 4 \ \ \cap \ -1 < Re(z+4-5i) \leq 6 \}$

d) $\left\{ z: \frac{Re(z)^2}{9} + \frac{Im(z)^2}{1} \leq 1 \ \cup \ \frac{Re(z)^2}{1} + \frac{Im(z)^2}{9} \leq 1 \ \cap Im(z) \leq 0 \ \right\}$

e) $\left\{ z: [Im(z-10i) > -5 \ \cup 2 \geq |z+1-0.5i|] \cap \arg(z+2) = \frac{\pi}{2} \ \cap \arg(z+2) = \ \frac{\pi}{2} \right\}$

f) $\left\{ z: \left[|z| \geq 1 \ \cap \ \frac{Re(z)^2}{4} + \frac{Im(z)^2}{9} \leq 1 \right] \cap Re(z) \leq 0 \ \ \right\}$

g) $\{ z: |z| \geq 4 \ \cup \ -12 < Im(z-8i) < \ -4 \ \cup \ -12 < Re(z-8) < -4 \ \ \}$

h) $\{ z: 0 < Re(z+2-5i) < 4 \ \cap 2 \leq Im(z+4i) \leq 6 \ \cup z \leq |z|: Re(z)Im(z) > 0 \ \}$

i) $\{z: |z| < 3 \ \cap \ -2 \leq Re(z-1-2i) \leq 1 \ \cap Im(z+6i) < 2\}$

j) $\{z: Im(z-4i) < 2 \ \cup \ -1 < Re(z+1+2i) \leq 6 \ \cup 3 \leq |z+2-4i| < 4\}$

k) $\{z: Im(z+3i) > 3 \ \cup z > 3\}$

l) $\{z: Im(z+7i) > 2 \ \cup |z-2| \geq 1 \}$

m) $\{z: Im(z+4i) > -2 \ \cap Im(z+1+i) > 0 \ \cap Re[z(z-1)] > 0 \}$

n) $\{z: |z| \geq 2 \ \cap \ Re(z+3) \leq 2 \ \cap \ Im(z-1) > 0\}$

WEEK 16

DEMOIVRE'S THEOREM

Abraham de Moivre (1667 – 1754) was a French mathematician who moved to England. It was here where is associated with Newton and Halley and became a private teacher of mathematics. One of his most predominant contributions was to the field of complex numbers.

De Moivre's theorem relates to finding powers of complex numbers.

Before we learn the shortcut – it is important to appreciate the pattern that develops through consecutive applications of multiplication. Start with the activity below.

WATCH THIS

Raising a complex number to a power by PatrickJMT (3 examples)

http://youtu.be/MO6qU3SCLbM

http://youtu.be/irwItP7PiQg

http://youtu.be/Srdh05cICiE

Play with this applet http://www.geogebratube.org/student/m5971

ACTIVITY 2

Using the multiplication rule for complex numbers in polar form, where $[r_1, \theta_1] \times [r_2, \theta_2] = [r_1 r_2,\ \theta_1 + \theta_2]$ to

a) investigate $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$ when n = 0, 1, 2, 3, … , 12

b) investigate $\left(3\cos\frac{\pi}{6} + 3i\sin\frac{\pi}{6}\right)^n$ when n = 0, 1, 2, 3, … ,6

What is the pattern you notice?

What you should see is that

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

In modulus argument form, this is

$$[r,\theta]^n = [r^n, n\theta]$$

This is de Moivre's theorem and this is true for any rational number n. (if you love proofs you could have a go at proving this using an induction method, although this is not formally part of the course)

There are many applications of de Moivre's theorem, one of them is in simplification of complex algebraic terms and another in the proof of trigonometric identities.

EXAMPLE 16

Prove that $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta\,(i\sin\theta) + 3\cos\theta\,(i\sin\theta)^2 + (i\sin\theta)^3 \\ &= \cos^3\theta + 3\,i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\end{aligned}$$

Comparing real parts of the equation we can get that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

EXAMPLE 17

Simplify the following expression:

$$\frac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta}$$

$$\begin{aligned}\frac{\cos 2\theta + i\sin 2\theta}{\cos 3\theta + i\sin 3\theta} &= \frac{(\cos\theta + i\sin\theta)^2}{(\cos\theta + i\sin\theta)^3}\\ &= \frac{1}{(\cos\theta + i\sin\theta)^1}\\ &= (\cos\theta + i\sin\theta)^{-1}\\ &= (\cos(-\theta) + i\sin(-\theta))\\ &= \cos\theta - i\sin\theta\end{aligned}$$

PRACTICE AND CONSOLIDATION 21

a) use de Moivre's theorem to prove the trig identities

i. $\sin 2\theta = 2\sin\theta\cos\theta$

ii. $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

b) If $z = \cos\theta + i\sin\theta$ then use de Moivre's theorem to show that

i. $z + \frac{1}{z} = 2\cos\theta$

ii. $z^n + \frac{1}{z^n} = 2\cos n\theta$

iii. $z^2 + \frac{1}{z^2} = 2\cos 2\theta$

c) Simplify the following

i. $\frac{\cos 5\theta + i\sin 5\theta}{\cos 2\theta - i\sin 2\theta}$

ii. $\frac{\cos\theta - i\sin\theta}{\cos 4\theta - i\sin 4\theta}$

PRACTICE AND CONSOLIDATION 22

Maths Quest Specialist Mathematics Chapter 3G Question 9

COMPLEX ROOTS OF UNITY

We can use de Moivre's theorem to find solutions to problems like $z^n = 1$. This category of problem is called a complex root of unity problem.

Let's start with an example

EXAMPLE 18

Find all the solutions to $z^3 = 1$.

Let $$z = [r, \theta]$$

then $$z^3 = [r, \theta]^3 = [r^3, 3\theta]$$

We can write the number 1 in modulus argument form like this: $$[1, 2n\pi]$$

So then we can write $$[r^3, 3\theta] = [1, 2n\pi]$$

Therefore by equating like parts we can see that $$r^3 = 1 \text{ and } 3\theta = 2n\pi$$ $$r = 1 \text{ and } \theta = \frac{2n\pi}{3}$$

We can then find the solutions by letting n=0,1,2, ..

If n=0 $$z_1 = [1,0] = 1$$

If n=1 $$z_2 = \left[1, \frac{2\pi}{3}\right] = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

If n=2 $$z_3 = \left[1, \frac{4\pi}{3}\right] = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

ACTIVITY 3

Using Example 19, what happens if you let n = 3, 4, and 5.

ACTIVITY 4

Plot the 3 distinct cubic roots of unity on an Argand diagram. What do you notice?

PRACTICE AND CONSOLIDATION 23

Use de Moivre's theorem to find all solutions to the following equations and plot the results on an Argand diagram. Get polar graph paper from the Learning commons to help with your graphs.

a) $z^2 = 1$
b) $z^3 = 1$
c) $z^4 = 1$
d) $z^5 = 1$
e) $z^6 = 1$
f) $z^3 = -1$
g) $z^4 = -1$

From the above 7 questions identify the pattern and draw a general conclusion.

h) $z^3 = 8$
i) $z^3 = i$
j) $16z^4 = 1$

Roots of unity occur in conjugate pairs and are evenly spaced around the plane. For $z^n = 1$, they are evenly spaced by an angle of $\frac{2\pi}{n}$

They are cyclic around the region, starting position equal to $\frac{2\pi}{n}$ if $z^n = 1$, and $\frac{\pi}{n}$ if $z^n = -1$

SQUARE ROOTS OF COMPLEX NUMBERS

EXAMPLE 10

Find the square root of $-8 + 6i$

Let $\sqrt{-8 + 6i} = a + bi$

$$-8 + 6i = (a + bi)^2$$

$$= a^2 - b^2 + i(2ab)$$

$$\therefore\ a^2 - b^2 = -8\ \&\ 2ab = 6$$

$$2ab = 6$$

$$b = \frac{3}{a}$$

So

$$a^2 - b^2 = -8$$

$$a^2 - \left(\frac{3}{a}\right)^2 = -8$$

$$a^2 - \frac{9}{a^2} = -8$$

$$a^4 - 9 = -8a^2$$

$$a^4 + 8a^2 - 9 = 0$$

$$(a^2 + 9)(a^2 - 1) = 0$$

From this we have that either

$$(a^2 + 9) = 0$$

$$a^2 = \pm\sqrt{-9}$$

$a = \pm 3i$ but by definition – both a and b are $\in \mathbb{R}$, so we can discount this answer

So

$$(a^2 - 1) = 0$$

$$a^2 = \pm\sqrt{1}$$

$$a = \pm 1$$

If $a = \pm 1$ then

$$b = \frac{3}{a} = \frac{3}{\pm 1} = \pm 3$$

So $\sqrt{-8 + 6i} = 1 + 3i$

(the convention we use here is that the sign ($\pm$) of $Re\left(\sqrt{z}\right) = \text{sign } (\pm)\ Re(z)$)

CHALLENGE 8

If $z^2$is known to be $3 - 4i$ what is z?

Now generalise to show what z is if $z^2$is $x + yi$

What does the CAS do for $\sqrt{3 - 4i}$? What do we need to know?

WATCH THIS

A series of 5 examples showing how to calculate the roots of complex numbers by PatrickJMT
http://youtu.be/HhlD7sX5Tp8
http://youtu.be/L0ZHlmOawQQ
http://youtu.be/CNCXNWTqxP4
http://youtu.be/VBl_mOWWAi4
http://youtu.be/_RtTdg3JYOE

PRACTICE AND CONSOLIDATION 24

Maths Quest Specialist Mathematics Chapter 3G Question 10, 11, 12, 13

WEEK 17

EULER'S THEOREM

Series Analysis

You may or may not have seen this before. What appears below is the series expansion of e^x, $\cos\theta$ and $\sin\theta$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

ACTIVITY 5

a) For each of the following values of θ, use the series for e^x with x replaced by $i\theta$ to calculate (to 4 d.p) the value of $e^{i\theta}$. Write your answers in in the form $a + bi$)

   i. $\theta = 0$
   ii. $\theta = 1$
   iii. $\theta = 2$
   iv. $\theta = -0.4$

b) Calculate $\cos\theta$ and $\sin\theta$ for each of the values in (a)

c) Find a connection between the values of $e^{i\theta}, \cos\theta \ \ and \ \ \sin\theta$ for each of the values of θ given in (a) and make up one other example to test your conjecture.

d) To prove this for all values of θ, write down the series expansions of $e^{i\theta}$, $\cos\theta \ \ and \ \ \sin\theta$ and show that $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Is known as Eulers Theorem.

It is an important results for our work in complex numbers. If z is any complex number then in polar form:

$$z = x + iy = rcos\,\theta + ri\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$re^{i\theta}$ is called the exponential form of a complex number – that's right – it's yet another form of a complex number!

Thus, $z^n = \left(re^{i\theta}\right)^n = r^n e^{ni\theta} = r^n e^{i(n\theta)}$ which can be shown to be de Moivre's Theorem.

An interesting result can be obtained from Euler's theorem if we substitute $\theta = \pi$. This gives

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0$$

$$\therefore e^{i\pi} + 1 = 0$$

This is often caller Eulers Equation, since it connects the five most famous numbers: 0, 1, π, e, i with a + and = sign.

PRACTICE AND CONSOLIDATION 25

Write each of the following complex numbers in the exponential form.

a) $2\left(\cos\frac{\pi}{3} + i\cos\frac{\pi}{3}\right)$

b) $\left[5, \frac{2\pi}{3}\right]$

c) $1 - i\sqrt{3}$

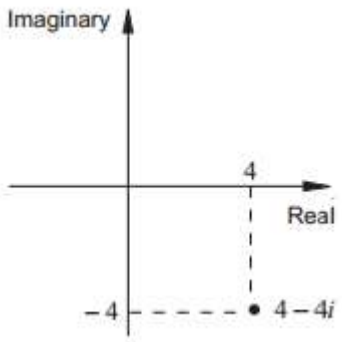
EXAMPLE 19

By first writing down the modulus and argument of the complex number $4 - 4i$, solve the equation

$z^5 = 4 - 4i$, expressing the answers in exponential form.

$$|4 - 4i| = \sqrt{4^2 + (-4)^2} = 4\sqrt{2}$$

A sketch of the number in the Argand plane can help identify the argument correctly.



$$4 - 4i = \left[4\sqrt{2}, -\frac{\pi}{4}\right]$$

Now we set about to solve $z^5 = 4 - 4i$

Let $z = [r, \theta]$, then $z^5 = [r^5, 5\theta]$

To obtain all 5 roots of the equation, we consider the argument to be $2n\pi - \frac{\pi}{4}$ where n is an integer.

Equating the results

$$[r^5, 5\theta] = \left[4\sqrt{2}, 2n\pi - \frac{\pi}{4}\right]$$

$$r^5 = 4\sqrt{2} \rightarrow r = \sqrt{2}$$

$$5\theta = 2n\pi - \frac{\pi}{4} \rightarrow \theta = (8n - 1)\frac{\pi}{20}$$

Remembering that the convention is that $-\pi < \theta < \pi$, choose appropriate values of n.

$$n = -2 \quad \rightarrow \quad \theta = \frac{-17\pi}{20}$$

$$n = -1 \quad \rightarrow \quad \theta = \frac{-9\pi}{20}$$

$$n = 0 \quad \rightarrow \quad \theta = \frac{-\pi}{20}$$

$$n = 1 \quad \rightarrow \quad \theta = \frac{7\pi}{20}$$

$$n = 2 \quad \rightarrow \quad \theta = \frac{15\pi}{20}$$

The solutions in exponential form are therefore

$$\sqrt{2}\, e^{\frac{-17\pi}{20}}, \quad \sqrt{2}\, e^{\frac{-9\pi}{20}} \quad , \sqrt{2}\, e^{\frac{-\pi}{20}}, \quad \sqrt{2}\, e^{\frac{7\pi}{20}}, \quad \sqrt{2}\, e^{\frac{15\pi}{20}}$$

ACTIVITY 6

Show that $1 + i$ is a root of the equation $z^4 = -4$ and find each of the other roots in the form $a + bi$ where a and b are real.

Plot the roots on an Argand diagram. By considering the diagonals, or otherwise, show that the points are at the vertices of a square. Calculate the area of the square.

ACTIVITY 7

Given that $k \neq 1$ and the roots of the quation $z^3 = k$ are α, β, γ use the substitution $z = \frac{x-2}{x+1}$ to obtain the roots of the equation $(x-2)^3 = k(x+1)^3$

PRACTICE AND CONSOLIDATION 26

a) Find all solutions to the following equations using de Moivre's theorem, give your answer in polar form. Plot each set of roots on an Argand diagram and comment on the symmetry.

   i. $z^4 = 16$
   ii. $z^3 = -27i$
   iii. $z^5 = -1$

b) Find the cube roots of, give your answers in exponential form

   i. $1 + i$
   ii. $2i - 2$

c) Using your answers from (a) part (i), find the solutions to the equation, give your answers in the form $a + bi$

$$(x+1)^4 = 16(x-1)^4$$

d) Using your answers from (a) part (ii), find the solutions to the equation, give your answers in the form $a + bi$

$$1 + 27i(x+1)^3 = 0$$

e) Determine the four roots of the equation, then plot them on an Argand diagram.

$$(z-2)^4 + (z+1)^4 = 0$$