

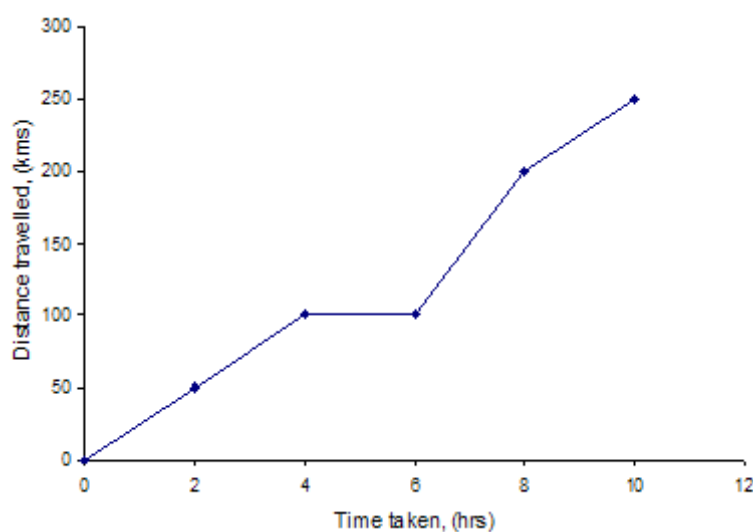
RATES OF CHANGE

RATES OF CHANGE (STRAIGHT LINES)

Rates of change are very important in mathematics. Take for example the 'speed' of a car. It is a measure of how far the car travels over a certain time, usually expressed in km/h. It literally means how many kilometres the car will travel per hour. It could also be expressed in km/s or m/s or mm/day. The important thing is that it is a measure of how much the distance changes for a certain change (increase) in time.

For instance, 100 km/h says that for every change in time of one hour, the distance travelled changes by 100 km.

If you graph the distance travelled by a car on the y-axis, and the time taken on the x-axis, for some journey you might get a graph like this:



Now, for this graph (and any graph of distance vs time), the gradient at any point represents the *speed* of the car at that time. For instance, between hours 0 and 4, the car is travelling at 25 km/h. Between hours 4 and 6, the gradient is 0, telling us the car is not moving. Then, between hours 6 and 8 the gradient is 50 km/h. Finally, between hours 8 and 10 the gradient is again 25 km/hr. In general, the slope of a graph can tell us the rate of change of the y-axis variable, *with respect to* the x-axis variable.

What this is saying is that the rate of change of a linear function can be measured or calculated, by working out the gradient, or slope, of the line. We know that for straight line (linear) functions, the gradient can be worked out using: $m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$.

These rates of change (the ones associated with linear functions) are called constant rates of change. As the value of the gradient calculated for these is always a constant (i.e. a number).

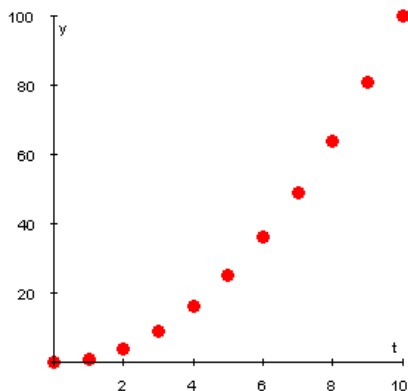
As rates of change are linked with the gradient of these straight lines we can see that:

- When a rate of change is positive, the graph goes up from left to right.
- When the rate of change is negative, the graph goes down from left to right.
- When the rate of change is zero, the graph is horizontal.

11 Math Methods 8B

Q1-3, Q6, Q8, Q12

NON CONSTANT RATES OF CHANGE (CURVED LINES)



Consider the graph on the left. It may have been created by plotting the results of an experiment. By displaying these measurements in the form of a graph, we can start to ask questions about how changes in y are occurring and assign more concrete meanings to the notions of rates of change. It is much more convenient to do this on a graph than a table of values.

11 Math Methods 8C

Q1, Q2, Q4, Q5

AVERAGE RATE OF CHANGE

In the figure below, we have identified a point P on the graph, and a second point, also on the graph which will serve as example. Note that a straight line has been drawn, connecting these two points

By how much has the value of y changed between the two points?

Over what period of time has this change occurred?

The ratio of these two values, i.e. (change in y)/(change in t) is called the average rate of change in y.

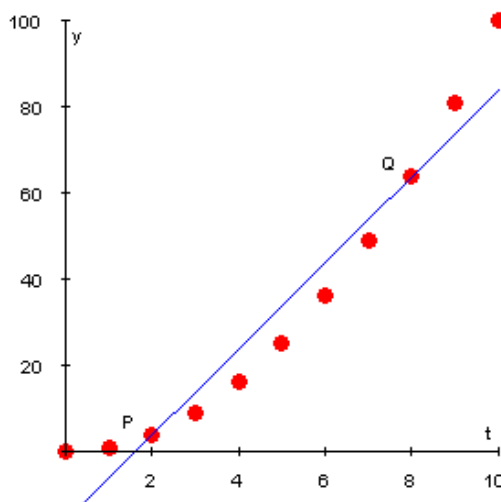
Recall we used this language before... change in y/change in t $\frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x}$. We are indeed finding the gradient of the line between the two points P and Q.

We call this the AVERAGE RATE OF CHANGE.

It is the rate of change of the line, connecting these two points.

The average rate of change is an important quantity which we can discuss without a graph. For instance, economists are interested in the average rate of change in unemployment over the past year.

However, once we draw our data points on a graph as above, we have an appealing geometrical interpretation of the average rate of change.



Average rate of change video with example: <http://youtu.be/rWt2TdXE9BA>

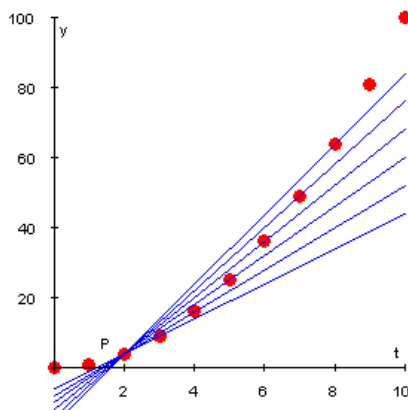
11 Math Methods 8D

Q1, Q2, Q3, Q7, Q8, Q9

INSTANTANEOUS RATES OF CHANGE

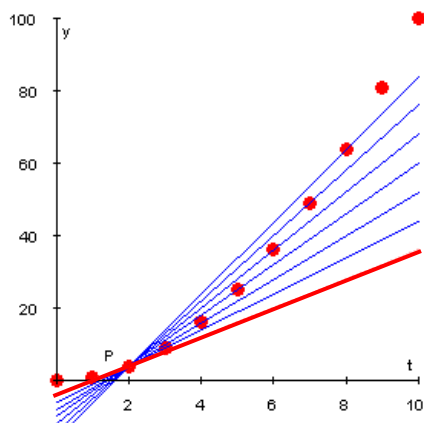
Of course, the value of the average rate that we computed depends not only on the point identified as P, but also on the second point Q on the graph. If we were to choose a different endpoint for the same calculation, i.e. a different point for Q, we would get a different average rate of change.

In the next picture, you can see what happens when the second point moves closer and closer to P. Notice that, as Q is chosen closer to P, the secant lines (shown in blue) have slopes that measure the change that take place very close to P.



Notice that the closer the secant lines converge towards P, a unique line through the point P occurs. That line is called the *tangent line*.

It has been drawn here in red, together with the secant lines, to show their relationship.



11 Math Methods Exercises 8E

Q1, Q2, Q5, Q6

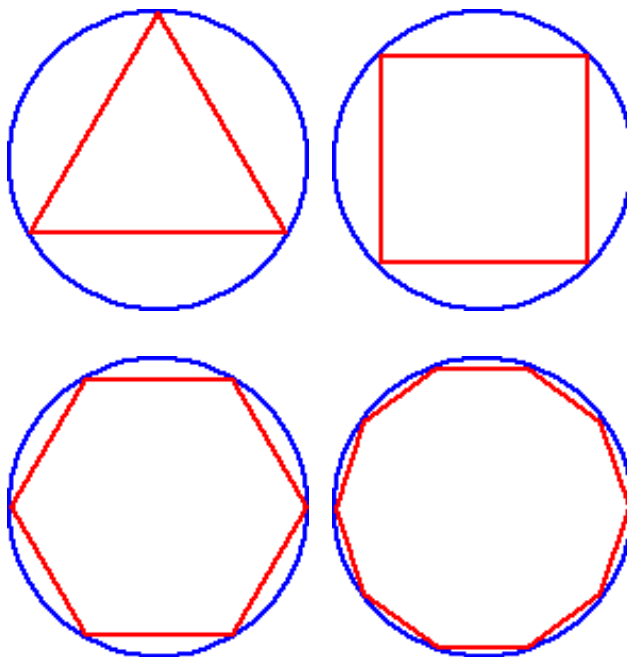
The tangent line represents a 'limiting process' in which the average rate of change is calculated over smaller intervals around P.

Since this rate of change is obtained by measuring the average rate of change close to P, we can think of it as measuring an *instantaneous* rate of change, or the rate of change at a point.

LIMITS

A GEOMETRIC EXAMPLE (FROM COOLMATH.COM)

Let's look at a polygon inscribed in a circle... If we increase the **number of sides** of the polygon, what can you say about the polygon with respect to the **circle**?



As the **number of sides** of the polygon **increase**, the **polygon** is getting closer and closer to becoming the **circle**!

If we refer to the **polygon** as an **n-gon**, where **n** is the **number of sides**, we can make some equivalent mathematical statements. (Each statement will get a bit more technical.)

- As **n** gets larger, the **n-gon** gets closer to being the **circle**.
- As **n** approaches infinity, the **n-gon** approaches the **circle**.
- The *limit* of the **n-gon**, as **n** goes to infinity, is the **circle**!

$$\lim_{n \rightarrow \infty} (\mathbf{n\text{-gon}}) = \mathbf{circle}$$

The **n-gon** never really gets to be the circle, but it will get very close! So close, in fact, that, for all practical purposes, it may as well be the circle. That's what limits are all about!

Archimedes used this idea (long before Calculus was even invented) to find the area of a circle before they had a value for pi. (They knew pi was the circumference divided by the diameter... but they didn't have calculators back then.)

NUMERICAL EXAMPLES

Let's look at some numerical examples, related to our work on functions...

$$f(x) = \frac{1}{x}$$

Write out a sequence of values for the counting numbers of x i.e. what is the value of the function $f(x)$ for values of x from 1, 2, 3, and so on...

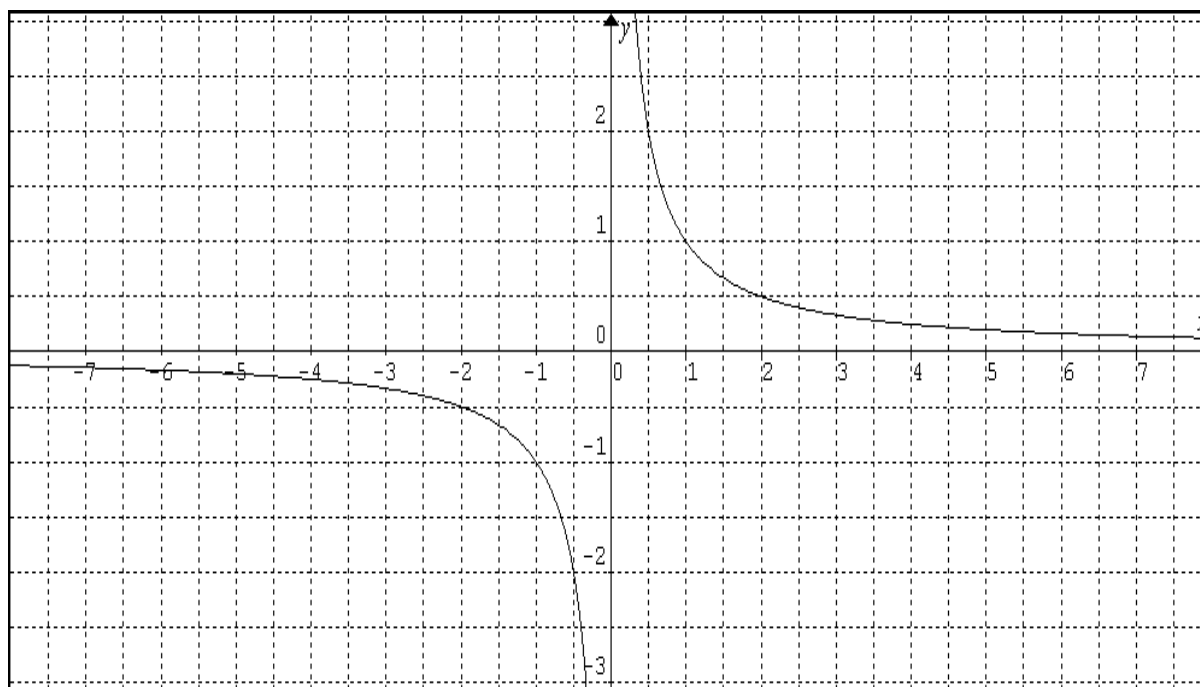
| | | | | | | | |
|------|-------------------|---------------|---------------|---------------------|----------------------|-----------------------|-------------------------|
| X | 1 | 2 | 3 | 4... | 50... | 600... | 10000... |
| F(x) | $\frac{1}{1} = 1$ | $\frac{1}{2}$ | $\frac{1}{3}$ | $\frac{1}{4} \dots$ | $\frac{1}{50} \dots$ | $\frac{1}{600} \dots$ | $\frac{1}{10000} \dots$ |

As x gets bigger and bigger, what is $f(x)$ approaching? It is approaching 0.

Writing this concept using mathematical symbols looks like this:

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

We actually need to be more specific. Let's further consider the same function, and look at the limits for extrema values and for asymptotic behaviour.



Remember what happened as x gets very, very, very large in the positive direction?

We write this using limits:

$$\lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

What about this one?

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

The limit as x approaches negative infinity of 1 over x is equal to 0.

What about the asymptote at $x=0$, we know the function does not exist at this point, so let's look at the function from both sides. $\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$ and $\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$. Because the limit as $x \rightarrow 0$ has a different limiting value from the two sides we state that the limit as $x \rightarrow 0$ does not exist.

For a limit to exist the function must approach the same value from both the negative and positive direction.

Let's look at another example using a rational function,

$$f(x) = \frac{x^2 - 1}{x - 1}$$

We know that $f(1)$ is not defined, yet as x approaches 1, $f(x)$ approaches what?

| $f(0.9)$ | $f(0.99)$ | $f(0.999)$ | $f(1.0)$ | $f(1.001)$ | $f(1.01)$ | $f(1.1)$ |
|----------|-----------|------------|---------------------------------------|------------|-----------|----------|
| 1.900 | 1.990 | 1.999 | $\Rightarrow \text{undef} \Leftarrow$ | 2.001 | 2.010 | 2.100 |

So from using the table of values we can see that as $x \rightarrow 1$, then $f(x)$ approaches 2. Thus, $f(x)$ can be made arbitrarily close to the limit of 2 just by making x sufficiently close to 1.

?? Have another look at the last function – can you sketch it? Is there a trick to it?

EVALUATING LIMITS

"Evaluating" means to find the value of





In the example above we said the limit was 2 because it looked like it was going to be. But sometimes that is not really adequate enough.

In fact there are many ways to get an accurate answer. Let's look at some:

1. JUST PUT THE VALUE IN

The first thing to try is just putting the value of the limit in, and see if it works (in other words substitution).

Let's try some examples:

| Example | | Substitute Value | Works? |
|--|---|---------------------|---|
| $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ |  | $(1-1)/(1-1) = 0/0$ |  |
| $\lim_{x \rightarrow 10} \frac{x}{2}$ |  | $10/2 = 5$ |  |

It didn't work with the first one (we knew that!), but the second example gave us a quick and easy answer.

2. FACTORS

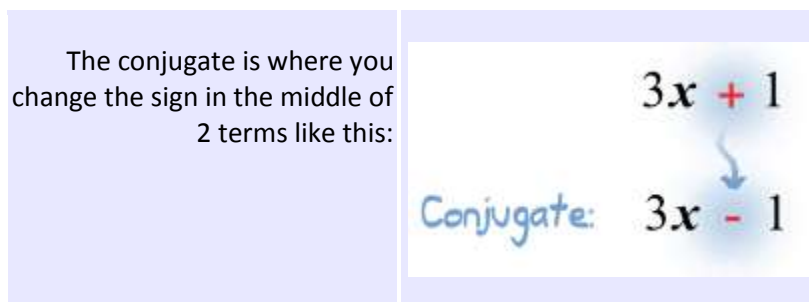
You can try factoring.

Example: $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$ By factoring $(x^2 - 1)$ into $(x - 1)(x + 1)$ we get:

$$f(x) = \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{x - 1} = \lim_{x \rightarrow 1} (x + 1) = 2$$

3. CONJUGATE

If it's a fraction, then multiplying top and bottom by a conjugate might help.



Here is an example where it will help you to find a limit:

$$\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4 - x}$$

Evaluating this at $x=4$ gives $0/0$, which is not a reasonable answer.

So, let's try some rearranging:

Multiply top and bottom by the conjugate of the top:

$$\frac{2 - \sqrt{x}}{4 - x} \times \frac{2 + \sqrt{x}}{2 + \sqrt{x}}$$

Simplify top using $(a + b)(a - b) = a^2 - b^2$:

$$\frac{2^2 - (\sqrt{x})^2}{(4 - x)(2 + \sqrt{x})}$$

Simplify top further:

$$\frac{(4 - x)}{(4 - x)(2 + \sqrt{x})}$$

Eliminate $(4-x)$ from top and bottom:

$$\frac{1}{2 + \sqrt{x}}$$

So, now we have:

$$\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{4 - x} = \lim_{x \rightarrow 4} \frac{1}{2 + \sqrt{x}} = \frac{1}{2 + \sqrt{4}} = \frac{1}{4}$$

Done!

4. INFINITE LIMITS

$$\frac{x^3 + 2x - 1}{6x^2}$$

If you can identify the overall Degree of the Function then you can find out whether the function's limit is 0, Infinity, -Infinity, or easily calculated from the coefficients.

If the degree on the numerator = the degree on the denominator, then you divide the coefficients of the terms with the largest exponent.

For example:

$$\lim_{x \rightarrow \infty} \frac{5x^3 - 2x^2 + x}{2x^3 - 10x^3 + 7} = \frac{5}{-10} = -\frac{1}{2}$$

If the degree on the numerator > the degree on the denominator,

then the limit is positive infinity ... or maybe negative infinity. **You need to look at the signs!**

You can work out the sign (positive or negative) by looking at the signs of the *terms with the largest exponent*, just like how we found the coefficients above:

For example this will go to positive infinity, because both ...

$$\frac{x^3 + 2x - 1}{6x^2}$$

- x^3 (the term with the largest exponent in the top) and
- $6x^2$ (the term with the largest exponent in the bottom)

... are positive.

$$\frac{-2x^2 + x}{5x - 3}$$

But this will head for negative infinity, because -2/5 is negative.

If the degree on the numerator < the degree on the denominator then the limit is 0.

11 Math Methods 9A Q4, Q7, Q8, Q9

11 Math Methods 9B Q1, Q2, Q5, Q10

SM Q7, Q8